Example

Find the exact value of

\[ \cos \left[ \tan^{-1} \sqrt{3} + \sin^{-1} \left( -\frac{2}{5} \right) \right]. \]

**SOLUTION:** The expression \( \cos \left[ \tan^{-1} \sqrt{3} + \sin^{-1} \left( -\frac{2}{5} \right) \right] \) is equivalent to \( \cos(\alpha + \beta) \), where

\[ \alpha = \tan^{-1} \sqrt{3}, \quad \text{and} \quad \beta = \sin^{-1} \left( -\frac{2}{5} \right). \]

From the definition of \( \tan^{-1} \), we know that \( \alpha \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) and, using the additional information that \( \tan(\alpha) = \sqrt{3} > 0 \), we may narrow the region further to place \( \alpha \in \left( 0, \frac{\pi}{2} \right) \).

Finally, the standard angle \( \pi/3 \) belongs to \( \left( 0, \frac{\pi}{2} \right) \) and satisfies \( \tan(\pi/3) = \sqrt{3} \), so it must be that \( \alpha = \pi/3 \).

From the definition of \( \sin^{-1} \), we know that \( \beta \) is in Q1 or QIV, but the further information that \( \sin(\beta) < 0 \) indicates that \( \beta \) can only be in QIV. So draw a triangle for \( \beta \) in QIV with side opposite to \( \beta \) of length -2, hypotenuse of length 5, and side adjacent to \( \beta \) of (positive) length to be determined. From the Pythagorean theorem, the side adjacent to \( \beta \) is of length \( \sqrt{5^2 - (-2)^2} = \sqrt{21} \). So

\[ \cos \beta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{21}}{5} \quad \text{and} \quad \sin \beta = \frac{\text{opp}}{\text{hyp}} = -\frac{2}{5}. \]

Using the sum formula for cosine, we have

\[
\begin{align*}
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
&= \cos \left( \frac{\pi}{3} \right) \cdot \left( \frac{\sqrt{21}}{5} \right) - \sin \left( \frac{\pi}{3} \right) \cdot \left( -\frac{2}{5} \right) \\
&= \frac{1}{2} \left( \frac{\sqrt{21}}{5} \right) - \sqrt{3} \cdot \left( -\frac{2}{5} \right) \\
&= \frac{\sqrt{21} + 2\sqrt{3}}{10}
\end{align*}
\]

This shows

\[ \cos \left[ \tan^{-1} \sqrt{3} + \sin^{-1} \left( -\frac{2}{5} \right) \right] = \frac{\sqrt{21} + 2\sqrt{3}}{10}. \]