Example

Consider the curve $y = f(x)$, where

$$f(x) = -x^3 + 27x - 6.$$ 

(a) Find the domain of $f$ and check for symmetries of the curve $y = f(x)$.

(b) Find the intervals where $f$ is increasing/decreasing, and where $f$ has local extrema.

(c) Find the intervals where $f$ is concave upward/downward, and the $x$-values of any points of inflection.

(d) Find any vertical or horizontal asymptotes for the curve $y = f(x)$.

(e) Find any $y$-intercepts of the curve.

(f) Use this information to sketch the graph of the curve.

Solution:

(a) The function $f(x)$ is a polynomial, so its domain is $(-\infty, \infty)$. To check for symmetries, we recall that the curve $y = f(x)$ is symmetric about the $y$-axis if $f(-x) = f(x)$, and symmetric about the origin if $f(-x) = -f(x)$, for all $x$ in the domain of $f$. For this particular function,

$$f(-x) = -(x^3) + 27(-x) - 6 = x^3 - 27x - 6$$

so $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. Thus the curve $y = f(x)$ is not symmetric about either the $y$-axis or the origin.

(b) We next find the critical numbers of $f$ making use of the fact that

$$f'(x) = -3x^2 + 27$$
$$= -3(x^2 - 9)$$
$$= -3(x - 3)(x + 3)$$

so that $f'(x) = 0$ at $x = -3, 3$. Since $f'(x)$ is defined for all values of $x$, the critical numbers of $f$ are

$$x = -3, 3.$$

These points divide the real line into intervals where $f'$ is of the same sign. A sign chart for $f'$ on these intervals follows:
\[
f'(x) = -3(x - 3)(x + 3)
\]

Since \( f \) is continuous, we may add finite endpoints to the intervals on which \( f \) increases/decreases, as follows:

- \( f \) is increasing for \( x \in [-3, 3] \),
- \( f \) is decreasing for \( x \in (-\infty, -3] \cup [3, \infty) \).

(c) Applying the First Derivative Test to the two critical numbers \( x = -3 \) and \( x = 3 \), we note that \( f \) changes from decreasing to increasing at the point \( x = -3 \), and \( f \) changes from increasing to decreasing at the point \( x = 3 \). It follows then that

- \( f \) has a local max at \( x = 3 \),
- \( f \) has a local min at \( x = -3 \).

The second derivative of \( f \) is given by

\[
f''(x) = -6x,
\]

from which it follows that \( f''(x) = 0 \) at \( x = 0 \). A sign chart for \( f'' \) follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>( f''(x) = -6x )</th>
<th>behavior of ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (-\infty, 0) )</td>
<td>+</td>
<td>concave upward,</td>
</tr>
<tr>
<td>( (0, \infty) )</td>
<td>-</td>
<td>concave downward,</td>
</tr>
</tbody>
</table>

so it follows that

- \( f \) is concave upward for \( x \in (-\infty, 0) \),
- \( f \) is concave downward for \( x \in (0, \infty) \),

and because the concavity changes at the point \( x = 0 \),

- \( f \) has a point of inflection at \( x = 0 \).

(d) To find the \( y \)-intercept of the curve (or the place where the curve crosses the \( y \)-axis), we note that this occurs when \( x = 0 \). Substituting \( x = 0 \) into \( f(x) = -x^3 + 27x - 6 \), it follows that \( f(0) = -6 \), or

- The \( y \)-intercept of the curve occurs at \( y = -6 \).
(e) The function $f(x)$ is defined and continuous for all values of $x$ so there are no vertical asymptotes. To check for horizontal asymptotes, the following limits are needed:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (-x^3 + 27x - 6) = \lim_{x \to -\infty} (-x^3) = \infty,$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (-x^3 + 27x - 6) = \lim_{x \to \infty} (-x^3) = -\infty,$$

where we have used the fact that the behavior of polynomials as $x \to \pm \infty$ is determined by the term in the polynomial with highest power on $x$. Because the above limits are not finite, the curve $y = f(x)$ has no horizontal asymptotes.

(f) To help with plotting the curve, we first find the $y$ coordinates of the $x$-values associated with local max, local min, and the point of inflection:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = -x^3 + 27x - 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-60</td>
</tr>
<tr>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
</tr>
</tbody>
</table>

Using the above information, the graph of $y = f(x) = -x^2 + 6x - 6$ is given below.