Example

Let $f$ be a *continuous* function with derivative given by

$$f'(x) = -3x^2 + 27.$$ 

(a) Find the intervals where $f$ is increasing and decreasing.

(b) Find the value(s) of $x$ where $f$ has local extrema.

(c) Find the interval(s) where $f$ is concave upward and/or concave downward.

(d) Find the $x$-values where $f$ has point(s) of inflection.

Solution:

(a) We first find the critical numbers of $f$ making use of the fact that

$$f'(x) = -3x^2 + 27 = -3(x^2 - 9) = -3(x - 3)(x + 3)$$

so that $f'(x) = 0$ at $x = -3, 3$. Since $f'(x)$ is defined for all values of $x$, the critical numbers of $f$ are

$$x = -3, 3.$$ 

These points divide the real line into intervals where $f'$ is of the same sign. A sign chart for $f'$ on these intervals follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>$x - 3$</th>
<th>$x + 3$</th>
<th>$f'(x) = -3(x - 3)(x + 3)$</th>
<th>behavior of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -3)$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>$(-3, 3)$</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>$(3, \infty)$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

Thus $f$ is increasing for $x \in (-3, 3)$, and $f$ is decreasing for $x \in (-\infty, -3) \cup (3, \infty)$.

But we know that $f$ is continuous (on all of $(\infty, \infty)$) so we can close the above intervals at finite endpoints. That is, we conclude that

- $f$ is increasing for $x \in [-3, 3]$,
- $f$ is decreasing for $x \in (-\infty, -3] \cup [3, \infty)$.

(b) Applying the First Derivative Test to the two critical numbers $x = -3$ and $x = 3$, we note that $f$ changes from decreasing to increasing at the point $x = -3$, and $f$ changes from increasing to decreasing at the point $x = 3$. It follows then that
• $f$ has a local max at $x = 3$,
• $f$ has a local min at $x = -3$.

Note that the Second Derivative Test could have also been used, with the same results.

(c) The second derivative of $f$ is given by

$$f''(x) = -6x,$$

from which it follows that $f''(x) = 0$ at $x = 0$. A sign chart for $f''$ follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>$f''(x) = -6x$</th>
<th>behavior of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>+</td>
<td>concave upward,</td>
</tr>
<tr>
<td>$(0, \infty)$</td>
<td>-</td>
<td>concave downward,</td>
</tr>
</tbody>
</table>

so it follows that

• $f$ is concave upward for $x \in (-\infty, 0)$,
• $f$ is concave downward for $x \in (0, \infty)$.

(d) From the above sign chart, the concavity changes at the point $x = 0$, so it follows that

• $f$ has a point of inflection at $x = 0$. 
