Example

Evaluate the limit
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]
for the function \( f(x) = x^3 \) and the point \( x = 2 \).

Solution: Using \( x = 2 \),
\[
f(x + h) = f(2 + h) = (2 + h)^3 = 2^3 + 3 \cdot 2^2 \cdot h + 3 \cdot 2 \cdot h^2 + h^3,
\]
i.e.,
\[
f(2 + h) = 8 + 12h + 6h^2 + h^3.
\]

It then follows that
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}
= \lim_{h \to 0} \frac{(8 + 12h + 6h^2 + h^3) - 8}{h}
= \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h},
\]
where we cannot simply evaluate the last expression at \( h = 0 \) because we will be dividing by zero. However, factoring leads to
\[
\lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h \to 0} \frac{(12 + 6h + h^2)h}{h}
= \lim_{h \to 0} (12 + 6h + h^2) \cdot \frac{h}{h}
= \lim_{h \to 0} (12 + 6h + h^2) = 12,
\]
where we have used the fact that
\[
\frac{h}{h} = 1, \quad \text{for all } h \neq 0.
\]