Example

A flexible cylinder has been built which can change its volume by changing its height or its radius, however, design constraints require that its height always be twice its radius. Find the rate of change of the volume of the cylinder when its radius is 3 feet and it is increasing at the rate of 2 ft/min.

Solution:

Let \( h(t) \) and \( r(t) \) denote the changing height and radius, respectively, of the cylinder, and let \( V(t) \) denote the changing volume of the cylinder.

We are given that
\[
h(t) = 2r(t),
\]
for all \( t > 0 \), and that for some instant \( t = t^* \) of time,
\[
r(t^*) = 3, \quad \text{and} \quad r'(t^*) = 2.
\]

The problem asks for the rate of change of the volume \( V(t) \) of the cylinder at this same \( t^* \); that is, we want
\[
V'(t^*).
\]

In order to obtain information about \( V'(t) \), it is useful to relate the quantities \( h(t) \), \( r(t) \) and \( V(t) \), since we are given information about \( r'(t) \). The changing volume of the cylinder is given by
\[
V(t) = \pi r^2(t) h(t) = \pi r^2(t) (2r(t)),
\]
or
\[
V(t) = 2\pi r^3(t),
\]
where we have used the fact that \( h(t) = 2r(t) \). Differentiating this last equation with respect to \( t \),
\[
V'(t) = 6\pi r^2(t) r'(t) = 6\pi r^2(t) \cdot 2,
\]
so at the time \( t^* \) of interest,
\[
V'(t^*) = 12\pi r^2(t^*) = 12\pi \cdot 3^2 = 108\pi \text{ ft}^3/\text{min}.
\]