Example

Find the linear approximation \( L(x) \) to the function

\[
f(x) = x^3 + 2x - 1
\]

at the point \( x = 2 \) and use it to approximate \( f \left( \frac{199}{100} \right) \).

Solution: Recall that for any differentiable function \( f \), the linear approximation to \( f \) at the point \( x = x_0 \) is given by

\[
L(x) = f(x_0) + f'(x_0)(x - x_0),
\]

where in this case,

\[
f(x) = x^3 + 2x - 1 \quad \text{and} \quad x_0 = 2.
\]

Differentiating, we have

\[
f'(x) = 3x^2 + 2,
\]

and since

\[
f(2) = 2^3 + 2 \cdot 2 - 1 = 11, \quad f'(2) = 3 \cdot 2^2 + 2 = 14,
\]

it follows that the linear approximation of \( f(x) \) at \( x_0 = 2 \) is

\[
L(x) = 11 + 14(x - 2).
\]

We can then use \( L(x) \) to find approximations of \( f(x) \) for \( x \) near 2. For example, for

\[
x = \frac{199}{100}
\]

it follows that

\[
L \left( \frac{199}{100} \right) = 11 + 14 \left( \frac{199}{100} - 2 \right) = 11 + 14 \left( -\frac{1}{100} \right) = \frac{1100 - 14}{100} = \frac{543}{50}
\]

which could be used as an approximation to \( f \left( \frac{199}{100} \right) \).

[Note that since \( x = 199/100 = 1.99 \), in fact for this example we can easily calculate the exact value of \( f(1.99) \),

\[
f(1.99) = 1.99^3 + 2 \cdot 1.99 - 1 = 10.8606.
\]
The linear approximation at $x = 199/100$ gives

$$L(199/100) = \frac{543}{50} = 10.86,$$

so in this case the linear approximation correctly determines the first 4 digits of $f(1.99)$. The use of decimals is for the purposes of illustration – decimal answers shouldn’t be given in WeBWorK.]