Example

Consider the function function 

\[ f(x) = x + 3x^{2/3}, \quad -1 \leq x \leq 1. \]

Determine the absolute maximum and minimum values of \( f \) and the \( x \)-values where these extrema occur.

Solution: To determine the critical numbers of \( f \), we compute \( f'(x) \):

\[
\begin{align*}
  f'(x) &= 1 + 3 \cdot \frac{2}{3} x^{-1/3} \\
  &= 1 + \frac{2}{x^{1/3}} \\
  &= \frac{x^{1/3} + 2}{x^{1/3}}.
\end{align*}
\]

Then \( f'(x) = 0 \) at those values of \( x \) for which

\[ x^{1/3} + 2 = 0, \quad \text{or} \quad x^{1/3} = -2 \quad \text{or} \quad x = (-2)^3 = -8. \]

In addition, \( f' \) is undefined at those values of \( x \) for which

\[ x^{1/3} = 0, \quad \text{or} \quad x = 0. \]

So the critical numbers are

\[ x = -8, \ 0, \]

but only \( x = 0 \) is in the interval \([-1, 1]\]

Evaluating \( f(x) \) at \( x = 0 \) and at the endpoints \( x = -1 \) and \( x = 1 \),

\[
\begin{align*}
  f(-1) &= -1 + 3(-1)^{2/3} = -1 + 3(-1)^2 = 2, \\
  f(1) &= 1 + 3 \cdot 1^{2/3} = 1 + 3 = 4, \\
  f(0) &= 0 + 3 \cdot 0^{2/3} = 0,
\end{align*}
\]

so it follows that the absolute maximum value of \( f \) is 4, which occurs at \( x = 1 \), and the absolute minimum value of \( f \) is 0 which occurs at \( x = 0 \).