Example

Let

\[ f(x) = \frac{x + 1}{x^2 + 3} \]

Find the intervals where \( f \) is increasing and/or decreasing and the \( x \)-values where \( f \) has local maxima or minima.

Solution: The domain of \( f \) is \((-\infty, \infty)\), since its denominator can never equal zero. To determine the critical numbers of \( f \), we compute

\[
 f'(x) = \frac{(x^2 + 3) - (x + 1)(2x)}{(x^2 + 3)^2} = \frac{x^2 + 3 - 2x^2 - 2x}{(x^2 + 3)^2} = \frac{-(x^2 + 2x - 3)}{(x^2 + 3)^2} = \frac{-(x + 3)(x - 1)}{(x^2 + 3)^2} \tag{1}
\]

and note that \( f'(x) \) is zero when the numerator in (1) is zero, i.e., at

\[ x = -3, 1. \]

Because the denominator of \( f' \) is always positive, there are no values of \( x \) for which \( f'(x) \) is undefined. Thus \( f \) has two critical numbers, at \( x = -3 \) and \( x = 1 \).

These critical numbers divide the domain of \( f \) into the open intervals \((-\infty, -3), (-3, 1) \) and \((1, \infty)\).

To determine whether \( f \) is increasing or decreasing on these intervals we make a sign chart with these intervals along the first column, and factors in the numerator and denominator of \( f' \) along the first row:

<table>
<thead>
<tr>
<th>interval</th>
<th>( x + 3 )</th>
<th>( x - 1 )</th>
<th>( (x^2 + 3)^2 )</th>
<th>( f'(x) = -(x - 3)(x + 1)/(x^2 + 3)^2 )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -3))</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td></td>
<td>decreasing</td>
</tr>
<tr>
<td>((-3, 1))</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td></td>
<td>increasing</td>
</tr>
<tr>
<td>((1, \infty))</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td></td>
<td>decreasing</td>
</tr>
</tbody>
</table>

Thus \( f \) is increasing on \((-3, 1)\) and it is decreasing on \((-\infty, -3) \cup (1, \infty)\).

The critical numbers \( x = -3 \) and \( x = 1 \) are the candidates local maxima/minima of \( f \):

- Because \( f \) changes from decreasing to increasing at the point \( x = -3 \), it follows that \( f \) has a local minimum at \((-3, f(-3)) = (-3, -1/6)\).

- The function \( f \) changes from increasing to decreasing at the point \( x = 1 \), so \( f \) has a local maximum at \((1, f(1)) = (1, 1/2)\).