Example

Let

\[ f'(x) = x^2 - 1, \]

be the derivative of a given function \( f \). Determine the open intervals where \( f \) is increasing and decreasing, and where \( f \) has local max/min. Also determine the open intervals where \( f \) is concave upward and concave downward, and where \( f \) has point(s) of inflection.

Solution: To determine the critical numbers of \( f \),

\[ f'(x) = x^2 - 1 = (x - 1)(x + 1) \]

so that \( f'(x) = 0 \) at \( x = -1, 1 \). Since \( f'(x) \) is defined for all values of \( x \), the critical numbers of \( f \) are

\[ x = -1, 1. \]

These points divide the real line into intervals where \( f' \) is of the same sign. A sign chart for \( f' \) on these intervals follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>( x - 1 )</th>
<th>( x + 1 )</th>
<th>( f'(x) = (x - 1)(x + 1) )</th>
<th>behavior of ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -1))</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>((1, \infty))</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
</tbody>
</table>

Thus:

- \( f \) is increasing on \((-\infty, -1)\) and \((1, \infty)\),
- \( f \) is decreasing on \((-1, 1)\).

Applying the First Derivative Test to the two critical numbers \( x = -1 \) and \( x = 1 \), we note that \( f \) changes from increasing to decreasing at the point \( x = -1 \), and \( f \) changes from decreasing to increasing at the point \( x = 1 \). It follows then that

- \( f \) has a local max at \( x = -1 \),
- \( f \) has a local min at \( x = 1 \).

Now taking the second derivative of \( f \),

\[ f''(x) = \frac{d}{dx} (x^2 - 1) = 2x, \]

it follows that \( f''(x) = 0 \) at \( x = 0 \). Making a sign chart for the values of \( f'' \) on the open intervals \((-\infty, 0)\) and \((0, \infty)\),
so it follows that

- $f$ is concave upward on $(0, \infty)$,
- $f$ is concave downward on $(-\infty, 0)$,

and because the concavity changes at the point $x = 0$,

- $f$ has a point of inflection at $x = 0$.