Example

Consider the curve $y = f(x)$, where

$$f(x) = 4x^{2/5}.$$

(a) Find the domain of $f$.

(b) Find any $x$-intercepts or $y$-intercepts of the curve.

(c) Check for symmetries of the curve $y = f(x)$.

(d) Find any vertical or horizontal asymptotes for the curve $y = f(x)$.

(e) Find the intervals where $f$ is increasing/decreasing, and where $f$ has local extrema.

(f) Find the intervals where $f$ is concave upward/downward, and the $x$-values of any points of inflection.

(g) Use this information to sketch the graph of the curve.

Solution:

(a) The domain of the function $f$ is $(-\infty, \infty)$.

(b) The $y$-intercept is the $y$-value of the curve as it crosses the $y$-axis, or the value of $f(x) = 4x^{2/5}$ at $x = 0$, i.e.,

- the $y$-intercept is $y = 0$.

The $x$-intercept(s) are the $x$-values where the curve crosses the $x$-axis, or the values of $x$ satisfying $f(x) = 0$. Since $4x^{2/5} = 0$ can only happen for $x = 0$ we have

- the $x$-intercept is at $x = 0$.

(c) To check for symmetries, we recall that the curve $y = f(x)$ is symmetric about the $y$-axis if $f(-x) = f(x)$, and symmetric about the origin if $f(-x) = -f(x)$, for all $x$ in the domain of $f$. For this particular function,

$$f(-x) = 4(-x)^{2/5} = 4x^{2/5} = f(x),$$

so the curve is symmetric about the $y$-axis.
(d) The function \( f(x) = 4x^{2/5} \) is defined and continuous for all values of \( x \), so there are no vertical asymptotes. To check for horizontal asymptotes, the following limits are needed:

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 4x^{2/5} = \infty \\
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 4x^{2/5} = \infty
\]

The lack of finite limits as \( x \to \pm\infty \) means the the curve \( y = f(x) \) does not have any horizontal asymptotes.

(e) Since

\[
f'(x) = 4 \cdot \frac{2}{5} x^{-3/5} = \frac{8}{5x^{3/5}}
\]

it follows that \( f'(x) \) is undefined at \( x = 0 \) and that \( f'(x) \) is never zero. Therefore, there is only one critical number, at

\[
x = 0.
\]

A sign chart for \( f' \) follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>( f'(x) = \frac{8}{5x^{3/5}} )</th>
<th>behavior of ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 0))</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>((0, \infty))</td>
<td>+</td>
<td>increasing</td>
</tr>
</tbody>
</table>

Since \( f \) is continuous, we may include the finite endpoints of the intervals when giving intervals on which \( f \) increases or decreases:

- \( f \) is increasing for \( x \in [0, \infty) \),
- \( f \) is decreasing for \( x \in (-\infty, 0) \).

Applying the First Derivative Test to the critical number \( x = 0 \) and noting that \( f \) changes from decreasing to increasing at that point,

- \( f \) has a local min at \( x = 0 \).

(f) Taking the second derivative of \( f \),

\[
f''(x) = \frac{d}{dx} \left( \frac{8}{5} x^{-3/5} \right) = \frac{8}{5} \cdot \frac{-3}{5} x^{-8/5} = -\frac{24}{25x^{8/5}},
\]

so \( f''(x) \) is never zero, but \( f''(x) \) is undefined at \( x = 0 \). A sign chart for the values of \( f'' \) follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>( f''(x) = -\frac{24}{25x^{8/5}} )</th>
<th>behavior of ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 0))</td>
<td>-</td>
<td>concave downward</td>
</tr>
<tr>
<td>((0, \infty))</td>
<td>-</td>
<td>concave downward</td>
</tr>
</tbody>
</table>
so it follows that

- $f$ is concave downward for $x \in (-\infty, 0) \cup (0, \infty)$,

but

- $f$ has no inflection points.

(g) Finally, the $y$-value associated with the critical number $x = 0$ is $y = f(0) = 0$, so that the point $(0, 0)$ is the location on the curve of its local min. Note that since $f'(0)$ is undefined, there is no horizontal tangent at this point on the curve.

The graph of the curve is shown below: