Example

Let 

\[ f(x) = 2x - 3x^2 \]

(a) Find the average value of \( f(x) \), \( f_{\text{ave}} \), on the interval \([-1, 5]\).

(b) From The Mean Value Theorem for Integrals, we know that there is a \( c \) in 
\((-1, 5)\) such that \( f(c) \) equals this average value you just found. Find \( c \).

Solution:

(a) Recall that the average value of a function \( f(x) \) on the interval \([a, b]\) is given by

\[ f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx, \]

so in this case

\[ f_{\text{ave}} = \frac{1}{5 - (-1)} \int_{-1}^5 (2x - 3x^2) \, dx. \]

In order to evaluate the integral, the following formulas may be used:

\[ \int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}, \quad \int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}. \]

So using linearity properties of the definite integral, it follows that

\[ \int_{-1}^5 (2x - 3x^2) \, dx = 2 \int_{-1}^5 x \, dx - 3 \int_{-1}^5 x^2 \, dx \]
\[ = 2 \left( \frac{5^2}{2} - \frac{(-1)^2}{2} \right) - 3 \left( \frac{5^3}{3} - \frac{(-1)^3}{3} \right) \]
\[ = 2 \left( \frac{25 - 1}{2} \right) - 3 \left( \frac{125 - (-1)}{3} \right) \]
\[ = 24 - 126 \]
\[ = -102. \]

Thus the average value of \( f(x) = 2x - 3x^2 \) over the interval \([-1, 5]\) is given by

\[ f_{\text{ave}} = \frac{1}{5 - (-1)} \int_{-1}^5 (2x - 3x^2) \, dx = \frac{-102}{6} = -17. \]
(b)

\[ f(c) = -17 \quad \Rightarrow \quad 2c - 3c^2 = -17 \]
\[ \Rightarrow \quad -3c^2 + 2c + 17 = 0 \]
\[ \Rightarrow \quad c = \frac{-2 \pm \sqrt{(2)^2 - 4(-3)(17)}}{2(-3)} \]
\[ \Rightarrow \quad c = \frac{1 \pm \sqrt{52}}{3}. \]

The only value of \( c \) in the interval \((-1, 5)\) is \( c = \frac{1 + \sqrt{52}}{3} \).