Example

Evaluate the definite integral

\[ \int_0^{\pi^2/400} \frac{\tan^3(5\sqrt{t}) \sec^2(5\sqrt{t})}{\sqrt{t}} \, dt. \]

Solution: Since \( \sec^2 z \) is the derivative of \( \tan z \), we will make the substitution

\[ u = \tan \left(5\sqrt{t}\right), \]

and thus

\[ du = \sec^2(5\sqrt{t}) \cdot 5 \cdot \frac{1}{2} t^{-1/2} \, dt = \frac{5}{2} \cdot \frac{\sec^2(5\sqrt{t})}{\sqrt{t}} \, dt, \]

so that

\[ \frac{\sec^2(5\sqrt{t})}{\sqrt{t}} \, dt = \frac{2}{5} \, du. \]

The lower and upper limits of integration for \( t \) variable are

lower limit for \( t = 0 \), \quad upper limit for \( t = \frac{\pi^2}{400}. \)

To convert this to a corresponding range of integration for the \( u \)-variable, we note that

\[ u(0) = \tan \left(5\sqrt{0}\right) = 0, \quad u\left(\frac{\pi^2}{400}\right) = \tan \left(5\sqrt{\frac{\pi^2}{400}}\right) = \tan \left(\frac{5\pi}{20}\right) = \tan \frac{\pi}{4} = 1, \]

so that the lower and upper limits for the \( u \) variable are

lower limit for \( u = 0 \), \quad upper limit for \( u = 1. \)

Making the substitutions given above,

\[ \int_0^{\pi^2/400} \frac{\tan^3(5\sqrt{t}) \sec^2(5\sqrt{t})}{\sqrt{t}} \, dt = \int_0^1 u^3 \cdot \frac{2}{5} \, du \]

\[ = \frac{2}{5} \int_0^1 u^3 \, du \]

\[ = \frac{2}{5} \left[ \frac{u^4}{4} \right]_0^1 \]

\[ = \frac{1}{10} (1^4 - 0^4) \]

\[ = \frac{1}{10}. \]

which is the final answer. There is no need to return to the original variable of integration (in this case, \( t \)) for a definite integral problem.