Example

Find all vertical, horizontal, and oblique asymptotes of the curve $y = f(x)$, where

$$f(x) = \frac{4x^2 - 3}{2x + 1}.$$

Solution:

- **Vertical asymptotes:** The function $f$ is undefined at $x = -1/2$. Using the fact that

$$\lim_{x \to (-1/2)^-} (2x + 1) = 0^-, \quad \text{and} \quad \lim_{x \to (-1/2)^+} (2x + 1) = 0^+,$$

and that the numerator of $f(x)$ is negative at $x = -1/2$, it follows that

$$\lim_{x \to (-1/2)^-} \frac{4x^2 - 3}{2x + 1} = +\infty, \quad \text{and} \quad \lim_{x \to (-1/2)^+} \frac{4x^2 - 3}{2x + 1} = -\infty,$$

or that the curve $y = f(x)$ has a single vertical asymptote at the line $x = -1/2$.

- **Horizontal asymptotes:** Looking at the behavior of $f(x)$ as $x \to \pm\infty$,

$$\lim_{x \to \infty} \frac{4x^2 - 3}{2x + 1} = \lim_{x \to \infty} \frac{4x - (3/x)}{2 + (1/x)} = +\infty,$$

and

$$\lim_{x \to -\infty} \frac{4x^2 - 3}{2x + 1} = \lim_{x \to -\infty} \frac{4x - (3/x)}{2 + (1/x)} = -\infty,$$

so the curve does not approach a constant (i.e., a horizontal line) as $x \to \pm\infty$. Thus there are no horizontal asymptotes for this curve.

- **Oblique asymptotes:** An oblique asymptote for the curve $y = f(x)$ gives even more information about the way (shown above) that the curve $y = f(x)$ becomes unbounded as $x \to \pm\infty$. To determine such an asymptote, we rewrite the function $f(x) = \frac{4x^2 - 3}{2x + 1}$ using long division:

$$\begin{array}{c|cc}
2x & -1 \\
2x + 1 & 4x^2 & +0x & -3 \\
& 4x^2 & +2x \\
& & -2x & -3 \\
& & & -2x & -1 \\
& & & & -2 \\
\end{array}$$
Thus, \[
f(x) = \frac{4x^2 - 3}{2x + 1} = 2x - 1 - \frac{2}{2x + 1},
\]
where \[
\lim_{x \to \infty} \frac{2}{2x + 1} = 0, \quad \text{and} \quad \lim_{x \to -\infty} \frac{2}{2x + 1} = 0.
\]
So as \(x \to \pm \infty\), the part of \(y = f(x)\) that does not go to zero is the line \(y = 2x - 1\).

It follows that the curve \(y = f(x)\) approaches this (non-horizontal) line as \(x \to \pm \infty\), or that the line \(y = 2x - 1\) is an oblique asymptote for the curve.