Example

The length of a side of a cube was measured to be 4 cm with a possible error of up to 1/10 cm.

Use a differential to estimate the maximum error in the calculated volume of the cube.

*Solution:* The volume $V(x)$ of a cube with side $x$ is given by

$$V(x) = x^3 \text{ cm}^3,$$

and the increment of volume from $x$ cm to $x + h$ cm is given by

$$\Delta V(x) = (V(x + h) - V(x)) \text{ cm}^3.$$

This is the actual error in volume. The differential of volume at $x$ with increment $h$ is given by

$$dV(x) = V'(x) \cdot h \text{ cm}^3,$$

which is an approximation to $\Delta V(x)$ provided that $h$ is small.

In this example, $x = 4$ and $h = \pm 1/10$. So the actual error in volume is the increment of volume from 4 cm to $(4 \pm 1/10)$ cm, or

$$\Delta V(4) = (V(4 \pm 1/10) - V(4)) \text{ cm}^3,$$

and the differential at $x = 4$ with increment is

$$dV(4) = V'(4) \cdot (\pm 1/10) \text{ cm}^3.$$

Using the fact that

$$V'(x) = 3x^2,$$

so $V'(4) = 3 \cdot 4^2 = 48$

it follows that

$$dV(4) = 48 \cdot \left( \pm \frac{1}{10} \right) = \pm \frac{24}{5} \text{ cm}^3,$$

so the max error in volume may be estimated by

$$|dV(4)| = \frac{24}{5} \text{ cm}^3.$$