Example

Let

\[
f(x) = \frac{2}{3}x^3 - 2x^2 - 6x + 1,
\]

(a) What is the domain of \( f \)?

(b) List each of the critical points of \( f \) and classify each as a local maximum, local minimum, or neither.

(c) Evaluate \( \lim_{x \to -\infty} f(x) \) and \( \lim_{x \to \infty} f(x) \).

(d) Find the absolute maximum and minimum values of \( f \), and the \( x \)-values where these absolute extrema occur.

Solution:

(a) The function \( f \) is a polynomial, so it is defined for all values of \( x \). Thus the domain of \( f \) is \(( -\infty, \infty )\).

(b) The derivative of \( f \) is defined everywhere and satisfies

\[
f'(x) = \frac{2}{3} \cdot 3x^2 - 4x - 6
\]

\[
= 2x^2 - 4x - 6
\]

\[
= 2(x^2 - 2x - 3)
\]

\[
= 2(x - 3)(x + 1),
\]

so it follows that

\[
f'(x) = 0 \quad \text{at} \quad x = -1, 3,
\]

which are the critical points for \( f \).

To classify the critical points we note that

\[
f''(x) = 4x - 4,
\]

and thus

\[
f''(-1) = 4 \cdot (-1) - 4 = -8 < 0,
\]

\[
f''(3) = 4 \cdot (3) - 4 = 8 > 0.
\]

It then follows from the Second Derivative Test that \( f \) has a local maximum at \( x = 1 \) and a local minimum at \( x = 3 \).
(c) When taking the limit as $x \to \pm\infty$ of a polynomial, it is the term with the highest power on $x$ which determines the value of the limit. In this case, this term is $\frac{2}{3}x^3$, so

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{2}{3}x^3 = -\infty,$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2}{3}x^3 = \infty.$$ 

(d) The results in (c) mean that $f$ grows without bound in the positive direction as $x \to \infty$ and $f$ grows without bound in the negative direction as $x \to -\infty$. So $f$ cannot have an absolute max or min value.