Example

Consider the function

\[ f(x) = x + 3x^{2/3}, \quad -1 \leq x \leq 1. \]

(a) Find all \( x \)-values for which this function has an endpoint extreme value (i.e., an endpoint maximum or endpoint minimum).

(b) Determine the absolute maximum and minimum values of \( f \) and the \( x \)-values where these extrema occur.

Solution: To give answers to both (a) and (b), it is helpful to first determine the critical points of \( f \) on the given interval. Since

\[
\begin{align*}
    f'(x) &= 1 + 3 \cdot \frac{2}{3} x^{-1/3} \\
    &= 1 + \frac{2}{x^{1/3}} \\
    &= \frac{x^{1/3} + 2}{x^{1/3}},
\end{align*}
\]

it follows that \( f'(x) = 0 \) at those values of \( x \) for which

\[ x^{1/3} + 2 = 0, \quad \text{or} \quad x^{1/3} = -2 \quad \text{or} \quad x = (-2)^3 = -8. \]

In addition, \( f' \) is undefined at those values of \( x \) for which

\[ x^{1/3} = 0, \quad \text{or} \quad x = 0. \]

So the critical points are \( x = -8, 0 \), but \( x = -8 \) is not in the given interval \([-1, 1]\) so it will not be considered below.

(a) To determine whether the \( f \) has an endpoint maximum or endpoint minimum at the endpoints \( x = -1 \) or \( x = 1 \), it is useful to make a sign chart for \( f'(x) \) on the subintervals of the domain \([-1, 1]\) which are determined by the critical point \( x = 0 \). These subintervals are

\[ [-1, 0), \quad (0, 1]. \]

Using the important factors in \( f'(x) = \frac{x^{1/3} + 2}{x^{1/3}} \) as columns in the chart, we obtain the following:

<table>
<thead>
<tr>
<th>interval</th>
<th>( x^{1/3} + 2 )</th>
<th>( x^{1/3} )</th>
<th>( f'(x) = \frac{x^{1/3} + 2}{x^{1/3}} )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1, 0) )</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>(0, 1] )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
</tbody>
</table>
Because $f$ is decreasing on the interval $[-1, 0)$, it has an endpoint maximum at $x = -1$. Similarly, $f$ is increasing on $(0, 1]$, so $f$ has an endpoint maximum at $x = 1$.

(b) To determine absolute maximum and minimum values of $f$ on $[-1, 1]$, we need to evaluate $f(x)$ at the endpoints $x = -1$ and $x = 1$, and at all critical points in the interval $[-1, 1]$:

\[
\begin{align*}
  f(-1) &= -1 + 3(-1)^{2/3} = -1 + 3(-1)^2 = 2, \\
  f(1) &= 1 + 3 \cdot 1^{2/3} = 1 + 3 = 4, \\
  f(0) &= 0 + 3 \cdot 0^{2/3} = 0,
\end{align*}
\]

so the absolute maximum value of $f$ is 4, which occurs at the endpoint $x = 1$. The absolute minimum value of $f$ is 0 which occurs at the critical point $x = 0$. 