Example

Let \( f(x) = |2x - 6| + 5 \)

Determine the domain of \( f \) and its critical points. Then find the intervals where \( f'(x) > 0 \) and \( f'(x) < 0 \), and where \( f \) has local (relative) extrema.

Solution: The domain of \( f \) is \((-\infty, \infty)\), since \( f(x) \) is defined for any real value of \( x \).

When a function is defined using absolute values, it is generally easier to work with the function if we rewrite it using the fact that

\[
|x| = \begin{cases} 
- x, & x < 0 \\
 x, & x \geq 0
\end{cases}
\]

For the current example,

\[
f(x) = |2x - 6| + 5 = \begin{cases} 
-(2x - 6) + 5, & (2x - 6) < 0 \\
2x - 6 + 5, & 2x - 6 \geq 0
\end{cases} = \begin{cases} 
-2x + 11, & x < 3 \\
2x - 1, & x \geq 3
\end{cases}
\]

Then to determine the critical points of \( f \), we compute

\[
f'(x) = \begin{cases} 
-2, & x < 3 \\
2, & x > 3
\end{cases}
\]

and for the derivative of \( f \) at \( x = 3 \), we must use the definition of the derivative to write

\[
f'(3) = \lim_{h \to 0} \frac{|2(3 + h) - 6| + 5 - (|2 \cdot 3 - 6| + 5)}{h} = \lim_{h \to 0} \frac{|2h|}{h} = 2 \lim_{h \to 0} \frac{|h|}{h},
\]

for which the limit fails to exist. Thus \( f'(x) \) is undefined at \( x = 3 \) and, since \( f'(x) \) is never zero, there is only one critical point of \( f \) given by \( x = 3 \).

The critical point divides the domain of \( f \) into two open intervals \((-\infty, 3)\) and \((3, \infty)\), and from equation (1) it is clear that \( f' > 0 \) on \((3, \infty)\) and \( f' < 0 \) on \((-\infty, 3)\).

The critical point \( x = 3 \) is the only candidate for local/relative maxima/minima of \( f \):

- Because \( f \) changes from decreasing to increasing at the point \( x = 3 \), it follows that \( f \) has a local/relative minimum at \((3, f(3)) = (3, 5)\).