Example

Let 

\[ f(x) = x^4 - x^3 \]

Determine the domain of \( f \) and its critical points. Then use the Second Derivative Test to determine all local (relative) extrema of \( f \). If the Second Derivative Test fails for a given critical point, use the First Derivative Test.

Solution: The domain of \( f \) is \((-\infty, \infty)\), since it is a polynomial. To determine the critical points of \( f \), we compute 

\[ f'(x) = 4x^3 - 3x^2 = x^2(4x - 3) \]

and letting \( f'(x) = 0 \) we get the critical points of \( f \) which are:

\[ x = 0, \frac{3}{4}. \]

To use the Second Derivative Test, we compute

\[ f''(x) = 12x^2 - 6x = 6x(2x - 1). \]

Evaluating \( f''(x) \) at the two critical points, we have:

\[ f''(0) = 0 \text{ and } f'' \left( \frac{3}{4} \right) = 6 \cdot \frac{3}{4} \left( 2 \cdot \frac{3}{4} - 1 \right) = \frac{9}{4} > 0. \]

From the Second Derivative Test, we can conclude that \( \frac{3}{4} \) is a local minimum, but no conclusion can be drawn at \( x = 0 \) because \( f''(0) = 0 \). So we have to use the First Derivative Test at this critical point.

Since the two critical points divide the domain of \( f \) into the open intervals \((-\infty, 0)\), \( (0, \frac{3}{4}) \) and \( (\frac{3}{4}, \infty) \), and we only need to determine whether \( f \) is increasing or decreasing on the intervals around \( x = 0 \), so we make a sign chart with these two intervals \((-\infty, 0)\), \( (0, \frac{3}{4}) \) along the first column, and factors of \( f' \) along the first row:

<table>
<thead>
<tr>
<th>interval</th>
<th>( x^2 )</th>
<th>( 4x - 3 )</th>
<th>( f'(x) = x^2(4x - 3) )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 0))</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>( (0, \frac{3}{4}) )</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

Thus \( f' < 0 \) on \((-\infty, 0)\) and \( (0, \frac{3}{4}) \) so it follows from the continuity of \( f \) that the function is decreasing on the entire interval \((-\infty, \frac{3}{4})\).

Finally, because \( f \) decreases immediately before and after the point \( x = 0 \), it follows that \( f \) has no local/relative extrema at \( x = 0 \).