Example

Consider the curve \( y = f(x) \), where
\[
f(x) = -x^3 + 27x - 6.
\]

(a) Find the domain of \( f \) and check for symmetries of the curve \( y = f(x) \).

(b) Find the intervals where \( f \) is increasing/decreasing, and where \( f \) has local (relative) extrema.

(c) Find the intervals where \( f \) is concave up/down, and the \( x \)-values of any points of inflection.

(d) Find any vertical or horizontal asymptotes for the curve \( y = f(x) \).

(e) Find any \( y \)-intercepts of the curve.

(f) Use this information to sketch the graph of the curve.

Solution:

(a) The function \( f(x) \) is a polynomial, so its domain is \((-\infty, \infty)\). To check for symmetries, we recall that the curve \( y = f(x) \) is symmetric about the \( y \)-axis if \( f(-x) = f(x) \), and symmetric about the origin if \( f(-x) = -f(x) \), for all \( x \) in the domain of \( f \). For this particular function,
\[
f(-x) = -(-x)^3 + 27(-x) - 6 = x^3 - 27x - 6
\]
so \( f(-x) \neq f(x) \) and \( f(-x) \neq -f(x) \). Thus the curve \( y = f(x) \) is not symmetric about either the \( y \)-axis or the origin.

(b) We next find the critical points of \( f \) making use of the fact that
\[
f'(x) = -3x^2 + 27
\]
\[
= -3(x^2 - 9)
\]
\[
= -3(x - 3)(x + 3)
\]
so that \( f'(x) = 0 \) at \( x = -3, 3 \). Since \( f'(x) \) is defined for all values of \( x \), the critical points of \( f \) are \( x = -3, 3 \).

These points divide the real line into intervals where \( f' \) is of the same sign. A sign chart for \( f' \) on these intervals follows:
Since $f$ is continuous, we may add finite endpoints to the intervals on which $f$ increases/decreases, as follows:

- $f$ is increasing for $x \in [-3, 3]$,
- $f$ is decreasing for $x \in (-\infty, -3] \cup [3, \infty)$.

(c) Applying the First Derivative Test to the two critical points $x = -3$ and $x = 3$, we note that $f$ changes from decreasing to increasing at the point $x = -3$, and $f$ changes from increasing to decreasing at the point $x = 3$. It follows then that

- $f$ has a local (relative) max at $x = 3$,
- $f$ has a local (relative) min at $x = -3$.

The second derivative of $f$ is given by

$$f''(x) = -6x,$$

from which it follows that $f''(x) = 0$ at $x = 0$. A sign chart for $f''$ follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>$f''(x) = -6x$</th>
<th>behavior of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>+</td>
<td>concave up,</td>
</tr>
<tr>
<td>$(0, \infty)$</td>
<td>-</td>
<td>concave down,</td>
</tr>
</tbody>
</table>

so it follows that

- $f$ is concave up for $x \in (-\infty, 0)$,
- $f$ is concave down for $x \in (0, \infty)$,

and because the concavity changes at the point $x = 0$,

- $f$ has a point of inflection at $x = 0$.

(d) To find the $y$-intercept of the curve (or the place where the curve crosses the $y$-axis), we note that this occurs when $x = 0$. Substituting $x = 0$ into $f(x) = -x^3 + 27x - 6$, it follows that $f(0) = -6$, or

- The $y$-intercept of the curve occurs at $y = -6$. 

<table>
<thead>
<tr>
<th>interval</th>
<th>$x - 3$</th>
<th>$x + 3$</th>
<th>$f'(x) = -3(x - 3)(x + 3)$</th>
<th>behavior of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -3)$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>$(-3, 3)$</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>$(3, \infty)$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
</tbody>
</table>
(e) The function $f(x)$ is defined and continuous for all values of $x$ so there are no vertical asymptotes. To check for horizontal asymptotes, the following limits are needed:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (-x^3 + 27x - 6) = \lim_{x \to -\infty} (-x^3) = \infty,$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (-x^3 + 27x - 6) = \lim_{x \to \infty} (-x^3) = -\infty,$$

where we have used the fact that the behavior of polynomials as $x \to \pm \infty$ is determined by the term in the polynomial with highest power on $x$. Because the above limits are not finite, the curve $y = f(x)$ has no horizontal asymptotes.

(f) To help with plotting the curve, we first find the $y$ coordinates of the $x$-values associated with local max, local min, and the point of inflection:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = -x^3 + 27x - 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-60</td>
</tr>
<tr>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
</tr>
</tbody>
</table>

Using the above information, the graph of $y = f(x) = -x^2 + 6x - 6$ is given below.