Example

Consider the curve $y = f(x)$, where

$$f(x) = \frac{4x^2}{2x + 1}.$$

(a) Find the domain of $f$, and check for symmetries of the curve $y = f(x)$.

(b) Find the intervals where $f$ is increasing/decreasing, and where $f$ has local (relative) extrema.

(c) Find the intervals where $f$ is concave up/down, and the $x$-values of any points of inflection.

(d) Find any vertical, horizontal, or oblique asymptotes for the curve $y = f(x)$.

(e) Find any $x$-intercepts or $y$-intercepts of the curve.

(f) Use this information to sketch the graph of the curve.

Solution:

(a) The function $f$ is defined for all $x$ except $x = -1/2$, so the domain of $f$ is the set $(-\infty, -1/2) \cup (1/2, \infty)$. To check for symmetries, we recall that the curve $y = f(x)$ is symmetric about the $y$-axis if $f(-x) = f(x)$, and symmetric about the origin if $f(-x) = -f(x)$, for all $x$ in the domain of $f$. For this particular function,

$$f(-x) = \frac{4(-x)^2}{2(-x) + 1} = \frac{4x^2}{-2x + 1},$$

so $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. The curve is not symmetric about the $y$-axis or about the origin.

(b) Using the quotient rule,

$$f'(x) = \frac{(2x + 1)(8x) - (4x^2)(2)}{(2x + 1)^2} = \frac{16x^2 + 8x - 8x^2}{(2x + 1)^2} = \frac{8(x^2 + x)}{(2x + 1)^2} = \frac{8x(x + 1)}{(2x + 1)^2}$$

so that

$$f'(x) = \begin{cases} 0, & \text{at } 8x(x + 1) = 0, \text{ or } x = 0, -1 \\ \text{undefined}, & \text{at } (2x + 1)^2 = 0, \text{ or } x = -1/2. \end{cases}$$

It follows that both $x = 0$ and $x = 1$ are critical points, however $x = -1/2$ is not a critical point because it is not in the domain of $f$.

Using the critical points $x = 0, -1$ to divide the domain of $f$ into subintervals, we obtain the following sign chart for $f'$:
\[ f'(x) = \frac{8x(1 + x)}{(2x + 1)^2} \]

<table>
<thead>
<tr>
<th>interval</th>
<th>(2x)</th>
<th>(x + 1)</th>
<th>((2x + 1)^2)</th>
<th>(f'(x))</th>
<th>behavior of (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((−\infty, 1/2))</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>((1/2, 0))</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>((0, \infty))</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
</tbody>
</table>

Since \(f\) is defined and continuous everywhere except at \(x = -1/2\), it follows that

- \(f\) is increasing for \(x \in (\infty, -1] \cup [0, \infty)\),
- \(f\) is decreasing for \(x \in [-1, -1/2) \cup (-1/2, 0]\),

and, using the First Derivative Test,

- \(f\) has a local max at \(x = -1\)
- \(f\) has a local min at \(x = 0\).

(c) Using the quotient rule again,

\[
\begin{align*}
f''(x) &= \frac{(2x + 1)^2(16x + 8) - 8(2x^2 + x) \cdot 2(2x + 1) \cdot 2}{(2x + 1)^4} \\
&= \frac{(2x + 1)(16x + 8) - 32(x^2 + x)}{(2x + 1)^3} \\
&= \frac{32x^2 + 32x + 8 - 32x^2 - 21x}{(2x + 1)^3} \\
&= \frac{8}{(2x + 1)^3}.
\end{align*}
\]

But \(f''(x)\) is never zero, and \(f''(x)\) is only undefined at \(x = -1/2\), which is not in the domain of \(f\). So we use a sign chart to examine the sign of \(f''(x)\) on the two parts of the domain of \(f\):

<table>
<thead>
<tr>
<th>interval</th>
<th>(2x + 1)</th>
<th>(f''(x))</th>
<th>behavior of (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((−\infty, -1/2))</td>
<td>-</td>
<td>-</td>
<td>concave down</td>
</tr>
<tr>
<td>((-1/2, \infty))</td>
<td>+</td>
<td>+</td>
<td>concave up</td>
</tr>
</tbody>
</table>

It follows that

- \(f\) is concave up for \(x \in (-1/2, \infty)\),
- \(f\) is concave down for \(x \in (-\infty, -1/2]\),

but that

- \(f\) does not have a point of inflection
since the concavity changes at a location which is not on the curve $y = f(x)$.

(d) The $y$-intercept of the curve is where the curve crosses the $y$-axis (i.e., at $x = 0$), and since $f(0) = 0$,

- the $y$-intercept of the curve is $y = 0$.

The curve has $x$-intercepts for those $x$ for which $f(x) = 0$. Since

$$f(x) = \frac{4x^2}{2x + 1} = 0, \quad \text{at } x = 0,$$

it follows that

- there is one $x$-intercept for the curve, at $x = 0$.

(e) The determination of vertical, horizontal, or oblique asymptotes is next:

(i) Vertical asymptotes: The function $f$ is undefined at $x = -1/2$. Then since

$$\lim_{x \to (-1/2)^-} \frac{4x^2}{2x + 1} = -\infty, \quad \text{and} \quad \lim_{x \to (-1/2)^+} \frac{4x^2}{2x + 1} = +\infty,$$

the curve has a vertical asymptote at $x = -1/2$.

(ii) Horizontal asymptotes: Looking at the behavior of $f(x)$ as $x \to \pm \infty$,

$$\lim_{x \to \infty} \frac{4x^2}{2x + 1} = \lim_{x \to \infty} \frac{4x}{2 + (1/x)} = +\infty,$$

and

$$\lim_{x \to -\infty} \frac{4x^2}{2x + 1} = \lim_{x \to -\infty} \frac{4x}{2 + (1/x)} = -\infty,$$

so the curve does not approach a constant (i.e., a horizontal line) as $x \to \pm \infty$. Thus there are no horizontal asymptotes for this curve.

(iii) Oblique asymptotes: The degree of the polynomial in the numerator of $f(x)$ exceeds the degree of the polynomial in the denominator by one, so the curve has an oblique asymptote. To determine this asymptote, we rewrite the function $f(x) = \frac{4x^2}{2x + 1}$ using long division:

\[
\begin{array}{r|ccc}
& 2x & -1 \\
\hline
2x + 1 & 4x^2 & + 0x & + 0 \\
& 4x^2 & + 2x \\
\hline
& -2x & -1 \\
\end{array}
\]

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Thus,

\[ f(x) = \frac{4x^2}{2x+1} = 2x - 1 + \frac{1}{2x+1}, \]

where

\[ \lim_{x \to \infty} \frac{1}{2x+1} = 0, \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{2x+1} = 0. \]

So as \( x \to \pm \infty \), the part of \( y = f(x) \) that does not go to zero is the line

\[ y = 2x - 1, \]

which is an oblique asymptote for the curve.

It follows then that

- the curve \( y = f(x) \) has a vertical asymptote at \( x = -1/2 \) and an oblique asymptote given by \( y = 2x - 1 \), but no horizontal asymptote.

(f) Finally, the \( y \)-coordinates for the locations of local max/min of \( f \) are given as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = f(x) = \frac{4x^2}{2x+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

A sketch of the graph is given below: