Example

Find the point on the curve $y = \sqrt{x}$ which is closest to the point $(2, 0)$.

Solution: For a point on the curve, its coordinate is $(x, \sqrt{x})$.

Then we can write a formula from a point on the curve to the point $(2, 0)$ by using the Distance Formula:

$$D(x) = \sqrt{(x - 2)^2 + (\sqrt{x} - 0)^2} = \sqrt{x^2 - 3x + 4}$$

Note that the domain of $y = \sqrt{x}$ is $[0, \infty)$, so $x \in [0, \infty)$ in $D(x)$.

To find the shortest distance from the curve to the point, we need to find the absolute minimum value of $D(x)$. Then we compute

$$D'(x) = \frac{2x - 3}{2\sqrt{x^2 - 3x + 4}}.$$  

Obviously, $D'(x) = 0$ at $x = \frac{3}{2}$. Because $x^2 - 3x + 4$ is greater than 0 for all $x \in [0, \infty)$, so the denominator of $D'(x)$ is always positive, there are no values of $x$ for which $D'(x)$ is undefined. Thus $D(x)$ only has one critical point, $x = \frac{3}{2}$.

Since $D'(x) > 0$ on $\left(\frac{3}{2}, \infty\right)$ and is continuous at the endpoints, $D(x)$ is strictly increasing on $\left[\frac{3}{2}, \infty\right)$. Since $D'(x) < 0$ on $\left(0, \frac{3}{2}\right)$ and is continuous at the endpoints, $D(x)$ is strictly decreasing on $\left[0, \frac{3}{2}\right]$. Then $D(x)$ has a local minimum at $x = \frac{3}{2}$.

Because $D$ is strictly decreasing before $x = \frac{3}{2}$ and strictly increasing after $x = \frac{3}{2}$, it must be that the point $x = \frac{3}{2}$ is an absolute minimum. This means that the point $\left(\frac{3}{2}, \sqrt{\frac{3}{2}}\right)$ on the curve $y = \sqrt{x}$ is closest to the point $(2, 0)$. 


Note: For a problem like this one, the derivative calculations are easier if we minimize $D^2(x) = x^2 - 3x + 4$ rather than to minimize $D(x) = \sqrt{x^2 - 3x + 4}$. Since both functions $D^2(x)$ and $D(x)$ have their minimum values at the same point, the conclusion will be the same in either case.