Example

Estimate the area under the graph of

\[ y = f(x) = 5 - x^2 \]

from \( x = -2 \) to \( x = 1 \) using the areas of 3 rectangles of equal widths, with heights of the rectangles determined by the height of the curve at (a) left endpoints, and (b) right endpoints.

**Solution:** The exact area in question is shaded below.

The width of the equally-spaced rectangles is given by

\[ \Delta x = \frac{b - a}{n} \]

where in this case, \( a = -2 \), \( b = 1 \), and \( n = 3 \), so

\[ \Delta x = \frac{1 - (-2)}{3} = 1. \]

(a) **Left endpoints:** In this case, the left endpoints of the intervals are given by

\[ x_k = a + (k - 1)\Delta x = -2 + (k - 1), \quad k = 1, 2, 3, \]

or

\[ x_1 = -2, \ x_2 = -1, \ x_3 = 0. \]

The heights of the rectangles are given in the following table:
So the approximate area is given by

\[ f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x = (1 + 4 + 5) \cdot 1 = 10. \]

(b) **Right endpoints:**

The right endpoints of the intervals are given by

\[ x_k = a + k\Delta x = -2 + k, \quad k = 1, 2, 3, \]

or

\[ x_1 = -1, \quad x_2 = 0, \quad x_3 = 1. \]

The heights of the rectangles are given in the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x_k )</th>
<th>( f(x_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>( f(-1) = 5 - (-1)^2 = 4 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( f(0) = 5 - 0^2 = 5 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( f(1) = 5 - 1^2 = 4 )</td>
</tr>
</tbody>
</table>

So the approximate area is given by

\[ f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x = (4 + 5 + 4) \cdot 1 = 13. \]