Example

The goal of this problem is to overestimate (underestimate) the area under the graph of

\[ y = f(x) = 5 - x^2 \]

from \( x = -2 \) to \( x = 1 \) using the areas of 3 rectangles of equal widths, where (a) an overestimate of the area (“upper sum”) is to be found and (b) an underestimate of the area (“lower sum”) is to be found.

Solution: The exact area in question is shaded below.

The width of the equally-spaced rectangles is given by

\[ \Delta x = \frac{b - a}{n} \]

where in this case, \( a = -2 \), \( b = 1 \), and \( n = 3 \), so

\[ \Delta x = \frac{1 - (-2)}{3} = 1. \]

(a) Overestimate: For the overestimate of the area, we need to take \( x_k \) to be the value of \( x \) for which \( f(x) \) is the largest in the \( k \)th subinterval. That is, since \( f \) is increasing on the negative \( x \)-axis, we will take right endpoints for intervals on the negative \( x \)-axis; since \( f \) is decreasing on the positive \( x \)-axis, we will take the left endpoints for intervals on the positive \( x \)-axis. So

\[ x_1 = -1, \ x_2 = 0, \ \text{and} \ x_3 = 0. \]

The heights of the rectangles are given in the following table:
The overestimate of the area is given by

\[ f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x = (4 + 5 + 5) \cdot 1 = 14. \]

(b) *Underestimate:*

For the underestimate of the area, we need to take \( x_k \) to be the value of \( x \) for which \( f(x) \) is the smallest in the \( k \)th subinterval. That is, since \( f \) is increasing on the negative \( x \)-axis, we will take left endpoints for intervals on the negative \( x \)-axis; since \( f \) is decreasing on the positive \( x \)-axis, we will take the right endpoints for intervals on the positive \( x \)-axis. So

\[ x_1 = -2, \ x_2 = -1, \text{ and } x_3 = 1. \]

The heights of the rectangles are given in the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x_k )</th>
<th>( f(x_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td>( f(-2) = 5 - (-2)^2 = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>( f(-1) = 5 - (-1)^2 = 4 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( f(1) = 5 - 1^2 = 4 )</td>
</tr>
</tbody>
</table>

The underestimate of the area is given by

\[ f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x = (1 + 4 + 4) \cdot 1 = 9. \]