Example

Find the area under the curve \( y = f(x) = 100 - 3x^2 \), from \( x = 1 \) to \( x = 5 \), using \( n \) equal-width rectangles and right endpoints, and taking the limit as \( n \to \infty \).

**Solution:** The region is graphed below:

The steps for computing the shaded area are as follows:

(a) The interval \([1, 5]\) is divided into \( n \) equal subintervals of length \( \Delta x \), where
\[
\Delta x = \frac{5 - 1}{n} = \frac{4}{n} .
\]

(b) We will use the notation \( x_k \) to represent the right-hand endpoint of the \( k \)th subinterval. Since
\[
\begin{align*}
x_1 &= 1 + \Delta x = 1 + \frac{4}{n}, \\
x_2 &= 1 + 2\Delta x = 1 + 2 \cdot \frac{4}{n}, \\
x_3 &= 1 + 3\Delta x = 1 + 3 \cdot \frac{4}{n}, \\
\vdots & \quad \vdots \\
x_k &= 1 + k\Delta x = 1 + k \cdot \frac{4}{n},
\end{align*}
\]
so it follows that
\[
x_k = 1 + k \cdot \frac{4}{n}. \]

(c) The area is then given by
\[
\text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x,
\]
where, using the fact that \( f(x) = 100 - 3x^2 \) and the formula for \( x_k \) given above,
\[
f(x_k) = 100 - 3x_k^2 = 100 - 3 \left(1 + k \cdot \frac{4}{n}\right)^2 = 100 - 3 \left(1 + 2k \cdot \frac{4}{n} + k^2 \frac{16}{n^2}\right) = 97 - \frac{24}{n}k - \frac{48}{n^2}k^2,
\]
so it follows that
\[ f(x_k)\Delta x = \left(97 - \frac{24}{n}k - \frac{48}{n^2}k^2\right)\left(\frac{4}{n}\right). \]

(d) We wish to obtain a closed-form expression for
\[ \sum_{k=1}^{n} f(x_k)\Delta x, \]
i.e., a simpler expression which does not involve the summation symbol. Since
\[ \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} \left(97 - \frac{24}{n}k - \frac{48}{n^2}k^2\right)\left(\frac{4}{n}\right) \]
\[ = \frac{4}{n} \sum_{k=1}^{n} \left(97 - \frac{24}{n}k - \frac{48}{n^2}k^2\right) \]
\[ = \frac{388}{n} \sum_{k=1}^{n} 1 - \frac{96}{n^2} \sum_{k=1}^{n} k - \frac{192}{n^3} \sum_{k=1}^{n} k^2, \]
we may use standard summation formulas to simplify, leading to
\[ \sum_{k=1}^{n} f(x_k)\Delta x = \frac{388}{n} n - \frac{96}{n^2} \left(\frac{1}{2} n^2 + \frac{1}{2} n\right) - \frac{192}{n^3} \left(\frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n\right) \]
\[ = 388 - 48 - \frac{96}{n} - 64 - \frac{32}{n^2} \]
\[ = 276 - \frac{144}{n} - \frac{32}{n^2}, \]
a closed-form representation of the summation.

(e) Finally, the area is given by
\[ \text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x \]
\[ = \lim_{n \to \infty} \sum_{k=1}^{n} \left(276 - \frac{144}{n} - \frac{32}{n^2}\right) \]
\[ = 276. \]