Example

Let 
\[ f(x) = 2x - 3x^2 \]

(a) Find the average value of \( f(x) \) on the interval \([-1, 5]\),

(b) From the First Mean Value Theorem for Integrals, we know that there is a \( c \) in \((-1, 5)\) such that \( f(c) \) equals this average value you just found. Find \( c \).

Solution:

(a) Recall that the average value of a function \( f(x) \) on the interval \([a, b]\) is given by
\[
\frac{1}{b-a} \int_a^b f(x) \, dx,
\]
so in this case we seek the value of
\[
\frac{1}{5 - (-1)} \int_{-1}^5 (2x - 3x^2) \, dx.
\]
In order to evaluate the integral, the following formulas may be used:
\[
\int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}, \quad \int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}.
\]
So using linearity properties of the definite integral, it follows that
\[
\int_{-1}^5 (2x - 3x^2) \, dx = 2 \int_{-1}^5 x \, dx - 3 \int_{-1}^5 x^2 \, dx
\]
\[
= 2 \left( \frac{5^2}{2} - \frac{(-1)^2}{2} \right) - 3 \left( \frac{5^3}{3} - \frac{(-1)^3}{3} \right)
\]
\[
= 2 \left( \frac{25 - 1}{2} \right) - 3 \left( \frac{125 - (-1)}{3} \right)
\]
\[
= 24 - 126
\]
\[
= -102.
\]
Thus the average value of \( f(x) = 2x - 3x^2 \) over the interval \([-1, 5]\) is given by
\[
\frac{1}{5 - (-1)} \int_{-1}^5 (2x - 3x^2) \, dx = \frac{-102}{6} = -17.
\]
(b)

\[ f(c) = -17 \implies 2c - 3c^2 = -17 \]
\[ \implies -3c^2 + 2c + 17 = 0 \]
\[ \implies c = \frac{-2 \pm \sqrt{(2)^2 - 4(-3)(17)}}{2(-3)} \]
\[ \implies c = \frac{1 \pm \sqrt{52}}{3}. \]

The only value of \( c \) in the interval \((-1, 5)\) is \( c = \frac{1 + \sqrt{52}}{3} \).