Example

Before looking at a particular example, recall the formal definition of a limit:

We say \( \lim_{x \to x_0} f(x) = L \) if, for every number \( \epsilon > 0 \), we can find a number \( \delta > 0 \) so that

\[
0 < |x - x_0| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon.
\]

The idea is that, given a number \( \epsilon > 0 \), the value of \( \delta > 0 \) is something over which you have control in order to constrain \( x \) via the inequality

\[
0 < |x - x_0| < \delta, \quad (1)
\]

and that we wish to make an appropriate selection of \( \delta > 0 \) so that we can guarantee the desired or wanted inequality

\[
|f(x) - L| < \epsilon, \quad (2)
\]

holds for this value of \( \epsilon \).

The following steps can be used to verify a limit statement such as

\[
\lim_{x \to x_0} f(x) = L.
\]

- **Step #1**: Rewrite the inequality (1) that we are able to control (i.e., the inequality involving \( \delta \)) as an interval around \( x - x_0 \), for \( x \neq x_0 \).

- **Step #2**: Rewrite the inequality (2) that we want to obtain (i.e., the inequality involving \( \epsilon \)) as an interval around \( x - x_0 \), for \( x \neq x_0 \). This is often done as follows:
  
  (a) First, rewrite the inequality (2) that we want to obtain as an inequality centered at \( x \).
  
  (b) Second, subtract \( x_0 \) from each part of this inequality centered at \( x \), and then change the inequality into an interval around \( x - x_0 \).

- **Step #3**: For \( x \neq x_0 \), select the largest value of \( \delta > 0 \) for which the interval obtained in Step #1 equals (or is contained in) the interval obtained in Step #2.

Now consider a particular example:

**Problem**: Prove

\[
\lim_{x \to 2} (3 - 2x) = -1
\]

using the formal definition of a limit.
Solution: Given any number $\epsilon > 0$, we will pick a value of $\delta > 0$ in order to control the values of $x$ via inequality

$$0 < |x - 2| < \delta,$$  \hspace{1cm} (3)

so that we may obtain the desired or wanted inequality

$$|(3 - 2x) - (-1)| < \epsilon$$  \hspace{1cm} (4)

for those values of $x$.

• **Step #1** Rewrite the inequality (3) (involving $\delta$) that we can control as an interval around $x - 2$, for $x \neq 2$:

That is,

$$0 < |x - 2| < \delta \implies x - 2 \in (-\delta, \delta), \ x \neq 2.$$  

• **Step #2** Rewrite the inequality (4) (involving $\epsilon$) that we want to obtain as an interval around $x - 2$, for $x \neq 2$:

(a) First rewrite (4) as an inequality centered at $x$:

The following statements are equivalent:

$$|(3 - 2x) - (-1)| < \epsilon \iff |4 - 2x| < \epsilon \iff -\epsilon < 4 - 2x < \epsilon.$$  

Subtracting 4 from each part of the last inequality leads to

$$-\epsilon - 4 < -2x < \epsilon - 4$$

and dividing through by $(-2)$ (reversing the direction of the inequalities) gives

$$-\frac{\epsilon - 4}{-2} > x > -\frac{\epsilon}{-2}$$

equivalently,

$$\frac{\epsilon}{2} + 2 > x > -\frac{\epsilon}{2} + 2,$$  \hspace{1cm} or  \hspace{1cm} $$2 - \frac{\epsilon}{2} < x < 2 + \frac{\epsilon}{2}.$$  

(b) Convert this last inequality centered at $x$ to an inequality centered at $x - 2$, and then rewrite the inequality as an interval around $x - 2$:

Subtracting 2 from each part of the last inequality,

$$-\frac{\epsilon}{2} < x - 2 < \frac{\epsilon}{2} \implies x - 2 \in \left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right).$$

• **Step #3:** For $x \neq 2$, select the largest value of $\delta > 0$ for which the interval obtained in Step #1, equals (or is contained in) the interval from Step #2.

The value of $\delta = \epsilon/2$ is such that $(-\delta, \delta) = (-\epsilon/2, \epsilon/2)$.

The solution is $\delta = \epsilon/2$. (That is, given any $\epsilon > 0$, the value of $\delta = \epsilon/2$ is such that, if $0 < |x - 2| < \delta$, then $|(3 - 2x) - (-1)| < \epsilon$.)

**Note:** The WeBWorK form of this solution is: $\text{eps/2}$