Example

Use the Newton’s method to estimate the real solution of

\[ x^3 + 3x + 1 = 0, \]

starting from the initial estimate \( x_0 = 0 \). (Explain why this is a good choice for a starting point of the iteration.) Then find \( x_1 \) and \( x_2 \).

**Solution:** First, we wish to find a zero of the function

\[ f(x) = x^3 + 3x + 1, \]

which is a continuous function defined for \( x \in (-\infty, \infty) \). Since

\[
\begin{align*}
  f(-1) &= (-1)^3 + 3(-1) + 1 = -3 < 0 \\
  f(1) &= (1)^3 + 3 \cdot 1 + 1 = 5 > 0,
\end{align*}
\]

it follows from the Intermediate Value Theorem that \( f(x) \) crosses the \( x \)-axis somewhere between \(-1\) and 1. So \( x_0 = 0 \) is a reasonable starting point for the Newton’s iteration.

The iterates in the Newton’s method are given by

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \]

starting from an initial estimate \( x_0 \) of the desired root. For this example we have

\[
\begin{align*}
  f(x) &= x^3 + 3x + 1, \\
  f'(x) &= 3x^2 + 3,
\end{align*}
\]

so the iteration becomes

\[ x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}, \quad n = 0, 1, \ldots, \]

starting from \( x_0 = 0 \). Then

\[
\begin{align*}
  x_1 &= x_0 - \frac{x_0^3 + 3x_0 + 1}{3x_0^2 + 3} \\
  &= 0 - \frac{0^3 + 3 \cdot 0 + 1}{3 \cdot 0^2 + 3} \\
  &= -\frac{1}{3} \\
  &\approx -0.3333,
\end{align*}
\]
rounded to four decimal places. Further,

\[ x_2 = x_1 - \frac{x_1^3 + 3x_1 + 1}{3x_1^2 + 3} \]

\[ = -.3333 - \frac{(-.3333)^3 + 3(-.3333) + 1}{3(-.3333)^2 + 3} \]

\[ \approx -.3222, \]

also rounded to four decimal places.

Thus, next two iterates from Newton’s Method are

\[ x_1 = -.3333, \quad \text{and} \quad x_2 = -.3222. \]