Example

List the points of intersection of the curves

\[ r = \cos \theta \quad r = \sin(2\theta). \]

Find the points in polar coordinates and then convert to Cartesian coordinates.

Recall that the graph of \( r = \cos \theta \) is a circle and the graph of \( r = \sin(2\theta) \) is a four-leaved rose “butterfly”. A plot of the curves is given here.

![Graphs of \( r = \cos \theta \) and \( r = \sin(2\theta) \) for 0 \( \leq \) \( \theta \) \( \leq \) 2\( \pi \).](image)

Figure 1: Graphs of \( r = \cos \theta \) and \( r = \sin(2\theta) \) for 0 \( \leq \) \( \theta \) \( \leq \) 2\( \pi \).

We can see that the curves intersect at the origin \((x, y) = (0, 0)\). We attempt to find the other points of intersection algebraically by setting the two polar equations equal.

\[ \cos \theta = \sin(2\theta) \]

\[ \cos \theta = 2 \sin \theta \cos \theta \]

\[ \frac{1}{2} = \sin \theta \]

\[ \theta = \frac{\pi}{6}, \frac{5\pi}{6} \]

Evaluating the two curves at \( \theta = \frac{\pi}{6} \),

\[ \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3} \]
so both graphs contain the point \((r, \theta) = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)\). Evaluating the two curves at \(\theta = \frac{5\pi}{6}\),
\[
\cos \frac{5\pi}{6} = \frac{-\sqrt{3}}{2} = \sin \frac{5\pi}{3}
\]
so both graphs contain the point \((r, \theta) = \left(\frac{-\sqrt{3}}{2}, \frac{5\pi}{6}\right)\).

Converting to Cartesian coordinates using the relations \(x = r \cos \theta\) and \(y = r \sin \theta\), the points of intersection are
\[(0, 0), \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right), \left(\frac{3}{4}, \frac{-\sqrt{3}}{4}\right)\].