Example

Find the area shared by the oval limaçon \( r = 4 + 2 \sin \theta \) and the circle \( r = 4 \).

The graphs of the oval limaçon \( r = 4 + 2 \sin \theta \) and the circle \( r = 4 \) are shown below.

![Graph of limaçon and circle](image)

Figure 1: Graphs of \( r = 4 + 2 \sin \theta \) and \( r = 4 \) for \( 0 \leq \theta \leq 2\pi \).

We can determine the points of intersection by solving algebraically

\[
\begin{align*}
4 + 2 \sin \theta &= 4 \\
\sin \theta &= 0 \\
\theta &= k\pi,
\end{align*}
\]

for any integer \( k \).

To compute the shared area, sum the areas of the upper semi-circle above the x-axis which is traced out by \( r = 4 \) as \( \theta \) ranges from 0 to \( \pi \),
Figure 2: Graphs of $r = 4 + 2 \sin \theta$ and $r = 4$ for $0 \leq \theta \leq \pi$.

and the area bounded above by the x-axis bounded and below by the limaçon $r = 4 + 2 \sin \theta$, where this portion of the limaçon is traced out as $\theta$ ranges from $-\pi$ to 0.

Figure 3: Graphs of $r = 4 + 2 \sin \theta$ and $r = 4$ for $-\pi \leq \theta \leq 0$.

Therefore the area shared by the oval limaçon $r = 4 + 2 \sin \theta$ and the circle $r = 4$ is

\[
\text{Area } = \int_{0}^{\pi} \frac{1}{2} (4)^2 \, d\theta + \int_{-\pi}^{0} \frac{1}{2} (4 + 2 \sin \theta)^2 \, d\theta
\]

\[
= 8\pi + \frac{1}{2} \int_{-\pi}^{0} (4 + 2 \sin \theta)^2 \, d\theta
\]

\[
= 8\pi + \frac{1}{2} \int_{-\pi}^{0} 16 + 16 \sin \theta + 4 \left( \frac{1 - \cos(2\theta)}{2} \right) \, d\theta
\]

\[
= 8\pi + \frac{1}{2} \left[(16\theta - 16 \cos \theta + 2\theta - \sin(2\theta)) \right]_{-\pi}^{0}
\]

\[
= -16 + 17\pi.
\]