Example

Consider the sequence \( \{a_n\} \) where the \( n \)th term is given by

\[
a_n = 5 \left( 1 - \frac{1}{n^3} \right)^{4n}.
\]

Determine if the sequence converges or diverges. If the sequence converges, find its limit.

The \( n \)th term can be written as

\[
a_n = 5 \left( 1 - \frac{1}{n^3} \right)^{4n} = 5 \exp \left( \ln \left( 1 - \frac{1}{n^3} \right) \right)^{4n} = 5 \exp \left( 4n \ln \left( 1 - \frac{1}{n^3} \right) \right).
\]

Then

\[
\lim_{n \to \infty} a_n = 5 \lim_{n \to \infty} \exp \left( 4n \ln \left( 1 - \frac{1}{n^3} \right) \right).
\]

Recalling that we can take the limit into the exponent since \( e^x \) is a continuous function. Using l’Hopital’s rule,

\[
\lim_{n \to \infty} \frac{\ln \left( 1 - \frac{1}{n^3} \right)}{4n} = \lim_{n \to \infty} \frac{-3n^{-4}}{(1 - \frac{1}{n^3}) / \left( \frac{-1}{4n^2} \right)} = \lim_{n \to \infty} \frac{12}{(1 - \frac{1}{n^3}) n^2} = 0,
\]

therefore

\[
\lim_{n \to \infty} a_n = 5e^0 = 5.
\]