Example

If possible, use the Integral Test to determine whether the following the infinite series converges or diverges:

\[ \sum_{n=1}^{\infty} \frac{4}{n^2 \sqrt{3n}}. \]  

(1)

**Solution:** To use the Integral Test to determine convergence of this series, we define the function

\[ f(x) = \frac{4}{x^2 \sqrt{3x}} \]  

where clearly this \( f(x) \) is positive, continuous, and decreasing for \( x \in [1, \infty) \). So the Integral Test *can* be used to analyze convergence of the series in (2).

To apply the Integral Test, we look at convergence/divergence of the improper integral

\[ \int_1^{\infty} \frac{4}{x^2 \sqrt{3x}} \, dx = \int_1^{\infty} \frac{4}{\sqrt{3}} x^{-5/2} \, dx. \]

In this case,

\[
\begin{align*}
\int_1^{\infty} \frac{4}{\sqrt{3}} x^{-5/2} \, dx &= \frac{4}{\sqrt{3}} \lim_{b \to \infty} \int_1^b x^{-5/2} \, dx \\
&= \frac{4}{\sqrt{3}} \lim_{b \to \infty} \left( \frac{1}{-3/2} \right) x^{-3/2} \bigg|_1^b \\
&= -\frac{8\sqrt{3}}{9} \lim_{b \to \infty} (b^{-3/2} - 1) \\
&= \frac{8\sqrt{3}}{9},
\end{align*}
\]

so the improper integral converges.

Therefore, by applying the Integral Test to the given infinite series in (1), we are able to conclude that the series in (1) converges as well.