Example

Determine if the Comparison Test or Limit Comparison test can be used to determine if each of the series converges or diverges.

1.
\[ \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n - 1} \]

2.
\[ \sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n + 1}} \]

1. Consider the series \( \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n - 1} \).

Since the terms behave globally like \( \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \), we suspect that the series diverges.

To use the Limit Comparison test to show divergence, we compare to the divergent p-series \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \). Thus,

\[ \lim_{n \to \infty} \frac{\sqrt{n}}{n - 1} = \lim_{n \to \infty} \frac{n}{n - 1} = 1. \]

Therefore by the Limit Comparison test, both series diverge.

To apply the Comparison test to show divergence, we must bound \( \frac{\sqrt{n}}{n - 1} \) below by the terms of a divergent series.

To obtain a lower bound on \( \frac{1}{n - 1} \), we first find an upper bound. Notice that for all \( n \geq 2 \),

\[ n - 1 < n \Rightarrow \frac{1}{n - 1} > \frac{1}{n} \Rightarrow \frac{\sqrt{n}}{n - 1} > \frac{1}{n}. \]

The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is a divergent p-series, therefore by the Comparison Test, both series diverge.

2. Consider the series \( \sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n + 1}} \).

To apply the Comparison test, the terms in the series must all be non-negative. To apply the Limit Comparison test, all terms in the series must be positive. Since \( \sin(n) \) can be negative for some \( n \geq 1 \), neither of these tests may be applied.