Example

Use the Limit Comparison Test to determine if the series converges or diverges.

\[
\sum_{n=1}^{\infty} \left( \frac{2n^3 - n^2 + 3n + 1}{(n + 3)^5} \right)^2.
\]

We first note that \( \frac{n^3 - n^2 + 3n + 1}{(n + 3)^5} \) behaves globally like \( \frac{n^3}{n^5} = \frac{1}{n^2} \).

Now considering the square on each term of the series, we conclude that the terms of the series behave like \( \left( \frac{1}{n^2} \right)^2 = \frac{1}{n^4} \).

Applying the Limit Comparison Test with \( \frac{1}{n^4} \),

\[
\lim_{n \to \infty} \frac{\left( \frac{n^3 - n^2 + 3n + 1}{(n + 3)^5} \right)^2}{\frac{1}{n^4}} = \lim_{n \to \infty} \left( \frac{n^3 - n^2 + 3n + 1}{(n + 3)^5} \right)^2 \cdot \left( \frac{n^2}{1} \right)^2 \\
= \lim_{n \to \infty} \left( \frac{n^5 - n^4 + 3n^3 + n^2}{(n + 3)^5} \right)^2 \\
= 1,
\]

where we have used the fact that \( x^2 \) is a continuous function allowing us to bring the limit inside the square, and l’Hospital’s rule repeatedly.

Since the limit above is finite and the series \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) is a convergent \( p \)-series, we conclude that the original series converges as well.