Example

Use the Limit Comparison Test to determine if the series converges or diverges.

\[
\sum_{n=1}^{\infty} \frac{2^{3n}}{(5^n + 1)^2}.
\]

We first note that \( \frac{2^{3n}}{(5^n + 1)^2} \) behaves globally like \( \frac{(2^3)^n}{(5^n)^2} = \frac{8^n}{25^n} = \left( \frac{8}{25} \right)^n \).

Applying the Limit Comparison Test with \( \left( \frac{8}{25} \right)^n \),

\[
\lim_{n \to \infty} \frac{2^{3n}}{(5^n + 1)^2} \left( \frac{8}{25} \right)^n = \lim_{n \to \infty} \frac{2^{3n}}{(5^n + 1)^2} \cdot \frac{5^{2n}}{2^{3n}}
\]

\[
= \lim_{n \to \infty} \frac{5^{2n}}{(5^n + 1)^2}
\]

\[= 1,\]

where we have used l’Hopital’s multiple times.

Since the limit above is finite and the series \( \sum_{n=1}^{\infty} \left( \frac{8}{25} \right)^n \) is a convergent geometric series, we conclude that the original series converges as well.