Example

Determine if the series converges or diverges.

\[
\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n+1}.
\]

The given series is neither geometric nor is it a \( p \)-series. We illustrate the application of multiple tests to this series.

1. **nth term test for divergence.**

   Computing
   \[
   \lim_{n \to \infty} \frac{\sqrt{n}}{n+1} = 0,
   \]
   we find that this test is inconclusive.

2. **Integral Test.**

   The function \( f(x) = \frac{\sqrt{x}}{x+1} \) is continuous, positive, and decreasing, so we may apply the integral test. Making a trig substitution \( x = \tan^2 \theta \) \( dx = 2 \tan \theta \sec^2 \theta \ d\theta \) \((0 < \theta < \pi/2)\),

   \[
   \int_{2}^{\infty} \frac{\sqrt{x}}{x+1} \ dx = \lim_{b \to \infty} \int_{2}^{b} \frac{\sqrt{x}}{x+1} \ dx
   \]
   \[
   = \lim_{b \to \infty} \int_{\arctan \sqrt{2}}^{\arctan \sqrt{b}} \frac{\tan \theta}{\sec^2 \theta} \ 2 \tan \theta \sec^2 \theta \ d\theta
   \]
   \[
   = \lim_{b \to \infty} \int_{\arctan \sqrt{2}}^{\arctan \sqrt{b}} 2 \tan^2 \theta \ d\theta
   \]
   \[
   = \lim_{b \to \infty} \int_{\arctan \sqrt{2}}^{\arctan \sqrt{b}} 2(\sec^2 \theta - 1) \ d\theta
   \]
   \[
   = 2 \lim_{b \to \infty} \left( \tan \theta - x \right)\bigg|_{\arctan \sqrt{2}}^{\arctan \sqrt{b}}
   \]
   \[
   = 2 \lim_{b \to \infty} \left( \sqrt{b} - \arctan \sqrt{b} \right) - \left( \sqrt{2} - \arctan \sqrt{2} \right)
   \]
   \[
   = \infty.
   \]

Therefore the series diverges by the integral test.
3. **Limit Comparison Test.** Since the terms behave globally like $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, we suspect that the series diverges.

To use the Limit Comparison test to show divergence, we compare to the divergent p-series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$. Thus,

$$\lim_{n \to \infty} \frac{\sqrt{n}}{n+1} = \lim_{n \to \infty} \frac{n}{n+1} = 1.$$ 

Therefore by the Limit Comparison test, both series diverge.