Example

Use the ratio test to determine if the series converges or diverges.

\[ \sum_{n=1}^{\infty} \frac{8n^2(n + 3)}{\ln n}. \]

Find the limit of the ratio, \( L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \).

Computing the ratio

\[
\frac{a_{n+1}}{a_n} = \frac{8(n + 1)^2(n + 1 + 3)}{\ln(n + 1)} \cdot \frac{\ln n}{8n^2(n + 3)}
\]

\[
= \left( \frac{n + 1}{n} \right)^2 \cdot \frac{n + 1 + 3}{n + 3} \cdot \frac{\ln n}{\ln(n + 1)}
\]

\[
= \left( 1 + \frac{1}{n} \right)^2 \left( 1 + \frac{1}{n + 3} \right) \frac{\ln n}{\ln(n + 1)}.
\]

Take the limit of the product by taking the product of the limits (provided they are finite):

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^2 = 1,
\]

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{n + 3} \right) = 1,
\]

\[
\lim_{n \to \infty} \frac{\ln n}{\ln(n + 1)} = \lim_{n \to \infty} \frac{1/n}{1/(n + 1)} = 1.
\]

Therefore

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \cdot 1 \cdot 1 = 1.
\]

Thus the ratio test is inconclusive. We may use

- The nth term test for divergence.

\[
\lim_{n \to \infty} \frac{8(n^3 + 3n^2)}{\ln(n)} = 8 \lim_{n \to \infty} \frac{3n^2 + 6n}{1/n} = \infty.
\]

so the series diverges.

- The comparison test, compared to the divergent harmonic series \( \sum_{i=1}^{\infty} 1/n \).

Since \( \ln(n) < n \) for all \( n \geq 1 \), it follows that

\[
\frac{1}{\ln(n)} > \frac{1}{n} \Rightarrow \frac{8(n^3 + 3n^2)}{\ln(n)} > \frac{1}{n}.
\]

Therefore both series diverge.