Example

Find the interval of convergence for the series

\[ \sum_{k=1}^{\infty} \frac{(-1)^k(x - 7)^k}{k + 2}. \]

Determine whether endpoints (if any) should be included in the interval of convergence.

We first compute

\[ \bar{\lambda} = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| \]

\[ = \lim_{k \to \infty} \left| \frac{(-1)^{k+1}(x - 7)^{k+1}/(k + 3)}{(-1)^k(x - 7)^k/(k + 2)} \right| \]

\[ = \lim_{k \to \infty} \left| (-1) \cdot \frac{k + 2}{k + 3} \cdot (x - 7) \right| \]

\[ = |x - 7|, \]

and then determine the values of \( x \) for which \( \bar{\lambda} < 1 \). It follows that

\[ |x - 7| < 1 \iff -1 < x - 7 < 1 \iff 6 < x < 8, \]

thus the series converges on the interval \( (6, 8) \).

Testing the endpoints:

at \( x = 6 \), \[ \sum_{k=1}^{\infty} \frac{(-1)^k(6 - 7)^k}{k + 2} = \sum_{k=1}^{\infty} \frac{1}{k + 2}, \]

which is divergent by the integral test (since \( \int_{1}^{\infty} \frac{dx}{x + 2} = \int_{3}^{\infty} \frac{du}{u} \) diverges);

at \( x = 8 \), \[ \sum_{k=1}^{\infty} \frac{(-1)^k(8 - 7)^k}{k + 2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k + 2}, \]

which is a convergent alternating series (since the sequence \( \left\{ \frac{1}{k + 2} \right\} \) is a positive, decreasing sequence converging to zero).

Therefore, the interval of convergence is the interval \( (6, 8] \).