Example

The function \( f(x) = \frac{6x^2}{2 - x} \) may be represented by the power series \( \sum_{n=0}^{\infty} c_n x^n \).

(a) What are the first three nonzero terms of this power series, starting with the nonzero term which has lowest power on \( x \)? (Note that the first term may be a constant term, i.e., that the power on \( x \) is zero.)

(b) In the general expression of the power series \( \sum_{n=0}^{\infty} c_n x^n \) for \( f(x) = \frac{6x^2}{2 - x} \), what is \( c_n \)?

(c) What is the radius \( R \) of convergence for this power series?

Solution:

(a) First notice that

\[
\frac{1}{1 - r} = \sum_{n=0}^{\infty} r^n, \quad |r| < 1,
\]

and the function \( f(x) \) can be rewritten as

\[
\frac{6x^2}{2 - x} = \frac{6x^2}{2} \cdot \frac{1}{1 - \frac{x}{2}} = 3x^2 \cdot \frac{1}{1 - \frac{x}{2}}.
\]

Let \( r = \frac{x}{2} \), then

\[
\frac{1}{1 - \frac{x}{2}} = \sum_{n=0}^{\infty} \left( \frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^n}.
\]

Multiplying through by \( 3x^2 \),

\[
\frac{3x^2}{1 - \frac{x}{2}} = 3x^2 \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} 3 \cdot 2^n \cdot x^{n+2} = \sum_{n=2}^{\infty} \frac{3}{2^{n-2}} \cdot x^n = 3x^2 + \frac{3}{2}x^3 + \frac{3}{4}x^4 + \cdots. \quad (1)
\]

Therefore,

\[
1\text{st non-zero term} = 3x^2
\]

\[
2\text{nd non-zero term} = \frac{3}{2}x^3
\]

\[
3\text{rd non-zero term} = \frac{3}{4}x^4.
\]
(b) From (1), it is obvious to see that

\[ c_n = \begin{cases} 
0, & n = 0, 1, \\
\frac{3}{2^{n-2}}, & n \geq 2.
\end{cases} \]

(c) From part (a), we know that (1) converges when \( |r| < 1 \), i.e., \( \left| \frac{x}{2} \right| < 1 \). So \( |x| < 2 \).
Therefore, the radius \( R \) of convergence is equal to 2.