Example

Find the length $L$ of the curve given by

$$y = \frac{x^{3/2}}{3} - \sqrt{x}, \quad x \in [1, 9].$$

Solution: Using $f(x) = (1/3)x^{3/2} - x^{1/2}$,

$$[f'(x)]^2 = \left[\frac{1}{2} \left(x^{1/2} - x^{-1/2}\right)\right]^2 = \frac{1}{4} \left(x - 2 + x^{-1}\right), \quad (1)$$

so

$$L = \int_1^9 \frac{1}{2} \sqrt{1 + (1/4) \left(x - 2 + x^{-1}\right)} \, dx$$

$$= \frac{1}{2} \int_1^9 \sqrt{x + 2 + x^{-1}} \, dx \quad (2)$$

which at first glance seems to pose a difficult integration problem. In fact, the problem at this point could be made considerably simpler if we were able to write the positive quantity

$$x + 2 + x^{-1}, \quad (3)$$

as a perfect square, thus eliminating the square root entirely. To see how one might do this, we note the similarity of (3) to the right-hand side of (1) (when the factor of $1/4$ is not considered), i.e., to

$$x - 2 + x^{-1}, \quad (4)$$

where from (1) we know that

$$x - 2 + x^{-1} = \left(x^{1/2} - x^{-1/2}\right)^2. \quad (5)$$

This suggests that we can write (3) as a perfect square by simply changing a sign in the right-hand side of (5):

$$x + 2 + x^{-1} = \left(x^{1/2} + x^{-1/2}\right)^2.$$

Returning now to (2), the integration can be handled in a straightforward way,

$$L = \frac{1}{2} \int_1^9 \sqrt{(x^{1/2} + x^{-1/2})^2} \, dx$$

$$= \frac{1}{2} \int_1^9 (x^{1/2} + x^{-1/2}) \, dx$$

$$= \frac{1}{2} \left. \left(x^{3/2} + \frac{x^{1/2}}{1/2}\right) \right|_1^9$$

$$= \frac{32}{3},$$

where the positivity of $x$ has been used in (6).