Example

Evaluate: \[ \int \cos 2x \cos 9x \, dx \]

Recall that the product of \( \cos \alpha \) and \( \cos \beta \) appears in the addition formulas for the cosine function,

\[
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta, \quad \text{and} \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.
\]

The summation of these two equations leads to an expression for \( \cos \alpha \cos \beta \) in terms of the sum of cosines of two different arguments, giving us something that is easier to integrate. That is,

\[
\begin{align*}
\cos 9x \cos 2x + \sin 9x \sin 2x &= \cos 7x = \cos(9x - 2x) \\
\cos 9x \cos 2x - \sin 9x \sin 2x &= \cos 11x = \cos(9x + 2x),
\end{align*}
\]

and therefore

\[
\cos 9x \cos 2x = \frac{1}{2} \cos 7x + \cos 11x.
\]

It follows then that

\[
\begin{align*}
\int \cos 2x \cos 9x \, dx &= \int \frac{1}{2} (\cos 7x + \cos 11x) \, dx \\
&= \frac{1}{2} \int \cos 7x \, dx + \frac{1}{2} \int \cos 11x \, dx \\
&= \frac{1}{14} \sin 7x + \frac{1}{22} \sin 11x + C.
\end{align*}
\]