Example

Find the points on the curve

\[ x(t) = 5 - 2 \sin t, \quad y(t) = 2 + 5 \cos t \quad t \in (-\infty, \infty) \]

for which there is (i) a horizontal tangent, (ii) a vertical tangent.

Solution: The curve has at least one tangent line at each of its points because the derivatives

\[ x'(t) = -2 \cos t \quad \text{and} \quad y'(t) = -5 \sin t, \]

are never simultaneously zero for \( t \in (\infty, \infty) \).

Since the slope of the tangent line to the curve is given by

\[ \frac{dy}{dx} = \frac{y'(t)}{x'(t)}, \]

we can determine values of \( t \) associated with (i) horizontal tangents by setting \( y'(t) = 0 \), and (ii) vertical tangents by setting \( x'(t) = 0 \).

We know that \( y'(t) = 0 \) for \( t = n\pi, n = 0, \pm 1, \pm 2, \ldots \), so horizontal tangents occur at all points of the form \((x(n\pi), y(n\pi))\). But

\[ x(n\pi) = 5 - 2 \sin(n\pi) = 5 \]

and

\[ y(n\pi) = 2 + 5 \cos(n\pi) = \begin{cases} 7, & n \text{ even} \\ -3, & n \text{ odd} \end{cases}, \]

so horizontal tangents occur at points \((5, 7)\) and \((5, -3)\) on the curve.

Similarly, \( x'(t) = 0 \) occurs when \( t = \frac{\pi}{2} + n\pi, n = 0, \pm 1, \pm 2, \ldots \), so vertical tangents occur at all points of the form \((x(\frac{\pi}{2} + n\pi), y(\frac{\pi}{2} + n\pi))\). Since

\[ x\left(\frac{\pi}{2} + n\pi\right) = 5 - 2 \sin\left(\frac{\pi}{2} + n\pi\right) = \begin{cases} 3, & n \text{ even} \\ 7, & n \text{ odd} \end{cases} \]

and

\[ y\left(\frac{\pi}{2} + n\pi\right) = 2 + 5 \cos\left(\frac{\pi}{2} + n\pi\right) = 2, \]

it follows that vertical tangents occur at points \((3, 2)\) and \((7, 2)\) on the curve.