Example

Find the arc length \( L(C) \) of the cardioid \( C \) given by the polar equation
\[
r = 2 - 2 \cos \theta.
\]

Solution: The graph of this cardioid is given below,

and we know that the curve is traced precisely once for \( \theta \in [0, 2\pi) \). Using the arc length formula for equations \( r = f(\theta) \) in polar coordinates, for \( a \leq \theta \leq b \),
\[
L(C) = \int_a^b \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta = \int_a^b \sqrt{f(\theta)^2 + [f'(\theta)]^2} \, d\theta,
\]
we then have in the case of \( f(\theta) = 2 - 2 \cos \theta \), \( 0 \leq \theta \leq 2\pi \),
\[
L(C) = \int_0^{2\pi} \sqrt{(2 - 2 \cos \theta)^2 + [2 \sin \theta]^2} \, d\theta
\]
or using the symmetry of the curve
\[
L(C) = 2 \int_0^{\pi} \sqrt{(2 - 2 \cos \theta)^2 + [2 \sin \theta]^2} \, d\theta
\]
\[
= 2 \int_0^{\pi} \sqrt{4 - 8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta} \, d\theta
\]
\[
= 2 \int_0^{\pi} \sqrt{4 - 8 \cos \theta + 4} \, d\theta
\]
\[
= 4 \sqrt{2} \int_0^{\pi} \sqrt{1 - \cos \theta} \, d\theta.
\]
To evaluate the integral we multiply the integrand by \( 1 = \frac{\sqrt{1 + \cos(\theta)}}{\sqrt{1 + \cos(\theta)}} \) to obtain
\[
L(C) = 4 \sqrt{2} \int_0^{\pi} \sqrt{1 - \cos \theta} \cdot \frac{1 + \cos(\theta)}{\sqrt{1 + \cos(\theta)}} \, d\theta
\]
\[
= 4 \sqrt{2} \int_0^{\pi} \frac{\sin \theta}{\sqrt{1 + \cos \theta}} \, d\theta
\]
where we have used the fact that $\sqrt{1 - \cos^2 \theta} = \sin \theta \geq 0$ for $\theta \in [0, \pi]$. If we let $u = 1 + \cos \theta$ and $du = -\sin \theta$, then

$$L(C) = -4\sqrt{2} \int_2^0 u^{-1/2} du = 4\sqrt{2} \left(2u^{1/2}\right)^2 \bigg|_2^0 = 8 \cdot 2 = 16.$$

Thus the arc length of the cardioid $r = 2 - 2 \cos \theta$ is 16.