Example

Find the area inside oval limaçon

\[ r = 4 + 2 \sin \theta. \]

We consider two approaches to the problem.

1. **Without using symmetry.**
   The curve \( r = 4 + 2 \sin \theta \) is traced out entirely as \( \theta \) ranges from 0 to \( 2\pi \).

   Figure 1: Graph of \( r = 4 + 2 \sin \theta \) for \( 0 \leq \theta \leq 2\pi \).

   

For polar curves, the area of the region enclosed by \( r = f(\theta) \) for \( \alpha \leq \theta \leq \beta \) is given by

\[ \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta. \]

Therefore, the area enclosed by the oval limaçon \( r = 4 + 2 \sin \theta \) from \( 0 \leq \theta \leq 2\pi \) is

\[
\text{Area} = \int_{0}^{2\pi} \frac{1}{2} (4 + 2 \sin \theta)^2 \, d\theta \\
= \frac{1}{2} \int_{0}^{2\pi} 16 + 16 \sin \theta + 4 \sin^2 \theta \, d\theta \\
= \frac{1}{2} \int_{0}^{2\pi} 16 + 16 \sin \theta + 4 \left( \frac{1 - \sin(2\theta)}{2} \right) \, d\theta \\
= \frac{1}{2} \left[ 16\theta - 16 \cos \theta + 2\theta + \cos(2\theta) \right]_{0}^{2\pi} \\
= 18\pi.
\]
2. **Using symmetry.**

Notice that if the point \((r, \theta)\) lies on the graph of \(r = 4 + 2\sin \theta\), then \((r, \pi - \theta)\) lies on the graph since

\[
4 + 2\sin(\pi - \theta) = 4 + 2[\sin(\pi) \cos(\theta) - \cos(\pi) \sin(\theta)] = 4 + 2\sin(\theta) = r,
\]

thus the curve is symmetric about the y-axis. The graph can be constructed using symmetry from the values of \(\theta\) ranging from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\).

Moreover, the area enclosed by the oval limaçon \(r = 4 + 2\sin \theta\) from \(0 \leq \theta \leq 2\pi\) is just *twice* the area enclosed by the curve as \(\theta\) ranges from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\).

Figure 2: Graph of \(r = 4 + 2\sin \theta\) for \(-\pi/2 \leq \theta \leq \pi/2\).

Thus

\[
\text{Area} = \int_{-\pi/2}^{\pi/2} \frac{1}{2} (4 + 2\sin \theta)^2 \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} 16 + 16\sin \theta + 4\sin^2 \theta \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} 16 + 16\sin \theta + 4 \left(\frac{1 - \sin(2\theta)}{2}\right) \, d\theta
\]

\[
= (16\theta - 16\cos \theta + 2\theta + \cos(2\theta)) \bigg|_{-\pi/2}^{\pi/2}
\]

\[
= 18\pi.
\]