Example

Find the area inside one leaf of the lemniscate

\[ r^2 = 2 \sin \theta. \]

We consider two approaches to the problem.

1. **Without using symmetry.**

   The curve \( r^2 = 2 \sin \theta \) is traced out entirely as \( \theta \) ranges from 0 to \( \pi \).

   ![Graph of \( r^2 = 2 \sin \theta \) for \( 0 \leq \theta \leq \pi \).](image)

   Notice that the upper leaf of the lemniscate \( r^2 = 2 \sin \theta \) is traced out by the function \( r = \sqrt{2 \sin \theta} \) as \( \theta \) ranges from 0 to \( \pi \). Similarly, the lower leaf is traced out by the function \( r = -\sqrt{2 \sin \theta} \) as \( \theta \) ranges from 0 to \( \pi \).

   For polar curves, the area of the region enclosed by \( r = f(\theta) \) for \( \alpha \leq \theta \leq \beta \) is given by

   \[
   \text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta.
   \]

   Therefore the area enclosed by the one leaf of the lemniscate \( r^2 = 2 \sin \theta \) is

   \[
   \text{Area} = \frac{1}{2} \int_{0}^{\pi} (\sqrt{2 \sin \theta})^2 \, d\theta \quad (= \frac{1}{2} \int_{0}^{\pi} (-\sqrt{2 \sin \theta})^2 \, d\theta)
   \]

   \[
   = \int_{0}^{\pi} \sin \theta \, d\theta
   \]

   \[
   = -\cos(\theta) \bigg|_{0}^{\pi}
   \]

   \[
   = 2.
   \]
2. Using symmetry.
Note that if \((r, \theta)\) lies on the graph of \(r^2 = 2\sin \theta\), then \((r, \pi - \theta)\) lies on the graph since 
\[2\sin(\pi - \theta) = 2\sin \theta = r^2,\] so the curve is symmetric about the y-axis. (It can also be shown that the graph is symmetric about the x-axis and the origin).

Notice that the right half of the upper leaf of the lemniscate \(r^2 = 2\sin \theta\) is traced out by the function \(r = \sqrt{2\sin \theta}\) as \(\theta\) ranges from 0 to \(\pi/2\).

![Graph of \(r = \sqrt{2\sin \theta}\) for \(0 \leq \theta \leq \pi/2\).](image)

Using symmetry about y-axis, the area of the leaf is twice the area enclosed by this curve, namely

\[
\text{Area} = 2 \int_0^{\pi/2} \frac{1}{2} \left(\sqrt{2\sin \theta}\right)^2 d\theta \\
= 2 \int_0^{\pi/2} \sin \theta \, d\theta \\
= -2 \cos \theta \bigg|_0^{\pi/2} \\
= 2.
\]