Example

Consider the following series:
\[
5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} + \cdots.
\]

(a) Give the value of the kth term \(a_k\) that allows us to write the series as \(\sum_{k=0}^{\infty} a_k\).

(b) What is the sum of the series?

Solution:

(a) In order to determine \(a_k\), find a pattern among the terms of the series:
\[
5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} + \cdots = 5 \cdot \frac{1}{1} - 5 \cdot \frac{1}{2} + 5 \cdot \frac{1}{4} - 5 \cdot \frac{1}{8} + \cdots
\]
\[
= 5 \cdot \frac{1}{2^0} - 5 \cdot \frac{1}{2^1} + 5 \cdot \frac{1}{2^2} - 5 \cdot \frac{1}{2^3} + \cdots
\]
\[
= 5 \cdot (\frac{-1}{2})^0 + 5 \cdot (\frac{-1}{2})^1 + 5 \cdot (\frac{-1}{2})^2 + 5 \cdot (\frac{-1}{2})^3 + \cdots
\]

To write this summation as \(\sum_{k=0}^{\infty} a_k\) we thus need to identify
\[
a_0 = 5 \cdot (\frac{-1}{2})^0, \quad a_1 = 5 \cdot (\frac{-1}{2})^1, \quad a_2 = 5 \cdot (\frac{-1}{2})^2, \quad \ldots,
\]
which means that
\[
a_k = 5 \cdot (\frac{-1}{2})^k, \quad \text{for } k = 0, 1, 2, \ldots.
\]

(b) Note that this series
\[
\sum_{k=0}^{\infty} 5 \cdot (\frac{-1}{2})^k = 5 \sum_{k=0}^{\infty} (\frac{-1}{2})^k,
\]
and \(\sum_{k=0}^{\infty} (\frac{-1}{2})^k\) is in the form of a geometric series \(\sum_{k=0}^{\infty} x^k\), where \(x = \frac{-1}{2}\). Since \(|x| < 1\), this series must converge. Further, its sum is given by
\[
5 \cdot \frac{1}{1 - x} = 5 \cdot \frac{1}{1 - (\frac{-1}{2})} = \frac{5}{\frac{3}{2}} = \frac{10}{3}.
\]