Example

Use the Basic Comparison Test to determine if the series converges or diverges.

\[ \sum_{k=1}^{\infty} \frac{\ln k}{(k^3 + 1)} \]

To use the Basic Comparison test to show convergence, we need to find a convergent series \[ \sum_{k=1}^{\infty} b_k \text{ for which } \frac{\ln k}{(k^3 + 1)} \leq b_k \text{ for all } k \text{ sufficiently large.} \]

To use the Basic Comparison test to show divergence, we need to find a divergent series \[ \sum_{k=1}^{\infty} c_k \text{ for which } \frac{\ln k}{(k^3 + 1)} \geq c_k \text{ for all } k \text{ sufficiently large.} \]

Notice that the numerator grows more slowly than \( k \) and the denominator grows with \( k^3 \), so we suspect that the terms of the series will “behave like” \( \frac{k}{k^3} \) (or \( \frac{1}{k^2} \)) and that the series is likely convergent.

To find an upper bound \( b_k \), we bound the numerator above and the denominator below as follows:

For all \( k \geq 1 \),

\[ \ln k < k \]

and

\[ k^3 + 1 > k^3 \]

from which it follows that

\[ \frac{1}{k^3 + 1} < \frac{1}{k^3}. \]

Thus

\[ \frac{\ln k}{k^3 + 1} < \frac{k}{k^3} = \frac{1}{k^2}. \]

The series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is a convergent \( p \)-series, therefore by the Basic Comparison Test, both series converge.