Example

Use the ratio test to determine if the series converges or diverges.

\[ \sum_{k=2}^{\infty} \frac{8k^2(k + 3)}{\ln k}. \]

**Solution:** Find the limit of the ratio, \( \lambda = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}. \)

Computing the ratio

\[
\frac{a_{k+1}}{a_k} = \frac{8(k + 1)^2(k + 1 + 3)}{\ln(k + 1)} \cdot \frac{\ln k}{8k^2(k + 3)}
= \left( \frac{k + 1}{k} \right)^2 \cdot \frac{k + 1 + 3}{k + 3} \cdot \frac{\ln k}{\ln(k + 1)}
= \left( 1 + \frac{1}{k} \right)^2 \left( 1 + \frac{1}{k + 3} \right) \frac{\ln k}{\ln(k + 1)}.
\]

Take the limit of the product by taking the product of the limits (provided they are finite):

\[
\lim_{k \to \infty} \left( 1 + \frac{1}{k} \right)^2 = 1,
\]

\[
\lim_{k \to \infty} \left( 1 + \frac{1}{k + 3} \right) = 1,
\]

\[
\lim_{k \to \infty} \frac{\ln k}{\ln(k + 1)} = \lim_{k \to \infty} \frac{1/k}{1/(k + 1)} = 1.
\]

Therefore

\[
\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = 1 \cdot 1 \cdot 1 = 1.
\]

Thus the ratio test is inconclusive. We may use

- The kth term test for divergence.

\[
\lim_{k \to \infty} \frac{8(k^3 + 3k^2)}{\ln(k)} = 8 \lim_{k \to \infty} \frac{3k^2 + 6k}{1/k} = \infty.
\]

so the series diverges.

- The basic comparison test, compared to the divergent harmonic series \( \sum_{k=2}^{\infty} \frac{1}{k}. \)

Since \( \ln(k) < k \) for all \( k \geq 1 \), it follows that

\[
\frac{1}{\ln(k)} > \frac{1}{k} \Rightarrow \frac{8(k^3 + 3k^2)}{\ln(k)} > \frac{1}{k}.
\]

Therefore both series diverge.