Example

Find the 3rd order Taylor polynomial in $x$ for the function $f(x) = \sinh\left(\frac{x}{2}\right)$. Also find the general expression for the $k$th nonzero term of the Taylor series in $x$ for this function, for $k = 0, 1, \ldots$.

The 3rd order Taylor polynomial $P_3$ in $x$ of a function $f(x)$ is given by

$$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}.$$ 

First compute the appropriate derivatives. If $f(x) = \sinh\left(\frac{x}{2}\right)$, then

$$f'(x) = \cosh\left(\frac{x}{2}\right) \cdot \frac{1}{2}$$

$$f''(x) = \sinh\left(\frac{x}{2}\right) \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$f'''(x) = \cosh\left(\frac{x}{2}\right) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

using chain rule. Using the definitions

$$\sinh\left(\frac{x}{2}\right) = \frac{e^{x/2} - e^{-x/2}}{2} \quad \cos\left(\frac{x}{2}\right) = \frac{e^{x/2} + e^{-x/2}}{2},$$

and evaluating at $x = 0$ for use in the formula above

$$f(0) = 0 \quad f'(0) = \frac{1}{2} \quad f''(0) = 0 \quad f'''(0) = \frac{1}{8},$$

we have

$$P_3(x) = \frac{1}{2} \cdot x + \frac{1}{8} \cdot \frac{x^3}{3!}.$$ 

Notice that $f^{(k)}(0) = 0$ when $k$ is even and $f^{(k)}(0) = \frac{1}{2^k}$ when $k$ is odd, for $k = 0, 1, 2, \ldots$. It follows that the $k$th nonzero term of the Taylor series must have (odd) power $2k + 1$, for $k = 0, 1, \ldots$. That is, the $k$th nonzero term in the Taylor series is given by the $(2k + 1)$st term in the summation $\sum_{k=0}^\infty f^{(k)}(0)\frac{x^k}{k!}$, or by

$$f^{(2k+1)}(0) \cdot \frac{x^{2k+1}}{(2k+1)!} = \frac{1}{2^{2k+1}} \cdot \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for} \quad k = 0, 1, 2, \ldots.$$