Example

Use the Direct Comparison Test to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\ln n}{(n^3 + 1)}.$$ 

To use the Direct Comparison test to show convergence, we need to find a convergent series \( \sum_{n=1}^{\infty} b_n \) for which \( \frac{\ln n}{(n^3 + 1)} \leq b_n \) for all \( n \) sufficiently large.

To use the Direct Comparison test to show divergence, we need to find a divergent series \( \sum_{n=1}^{\infty} c_n \) for which \( \frac{\ln n}{(n^3 + 1)} \geq c_n \) for all \( n \) sufficiently large.

Notice that the numerator grows more slowly than \( n \) and the denominator grows with \( n^3 \), so we suspect that the terms of the series will “behave like” \( \frac{n}{n^3} \) (or \( \frac{1}{n^2} \)) and that the series is likely convergent.

To find an upper bound \( b_n \), we bound the numerator above and the denominator below as follows:

For all \( n \geq 1 \),

\[ \ln n < n \]

and

\[ n^3 + 1 > n^3 \]

from which it follows that

\[ \frac{1}{n^3 + 1} < \frac{1}{n^3}. \]

Thus

\[ \frac{\ln n}{n^3 + 1} < \frac{n}{n^3} = \frac{1}{n^2}. \]

The series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a convergent \( p \)-series, therefore by the Direct Comparison Test, both series converge.