Example

Find the Taylor polynomials of orders 0, 1, 2, and 3 for the function \( f(x) = \frac{1}{4-x} \) about the point \( x = 2 \). What is \( R_3(x) \), the remainder of order 3?

The Taylor polynomial \( P_n \) of order \( n \) of a function \( f(x) \) about the point \( x = a \) is given by

\[
P_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}.
\]

We need to determine \( P_n(x) \) for \( n = 0, 1, 2, \) and 3 with \( a = 2 \). Using the formula above, it follows that

\[
P_0(x) = f(2),
\]
\[
P_1(x) = f(2) + f'(2)(x-2),
\]
\[
P_2(x) = f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2!},
\]
\[
P_3(x) = f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2!} + f'''(2)\frac{(x-2)^3}{3!}.
\]

First compute the appropriate derivatives. If \( f(x) = \frac{1}{4-x} = (4-x)^{-1} \), then

\[
f'(x) = -(4-x)^{-2} \cdot (-1) = (4-x)^{-2} = \frac{1}{(4-x)^2}
\]

\[
f''(x) = (-2)(4-x)^{-3} \cdot (-1) = 2(4-x)^{-3} = \frac{2}{(4-x)^3}
\]

\[
f'''(x) = (-6)(4-x)^{-4} \cdot (-1) = 6(4-x)^{-4} = \frac{6}{(4-x)^4}
\]

using chain rule. Evaluating at \( x = 2 \) for use in the formula above,

\[
f(2) = \frac{1}{2}, \quad f'(2) = \frac{1}{4}, \quad f''(2) = \frac{1}{4}, \quad f'''(2) = \frac{3}{8}.
\]
Thus,

\[ P_0(x) = \frac{1}{2}, \]

\[ P_1(x) = \frac{1}{2} + \frac{x - 2}{4}, \]

\[ P_2(x) = \frac{1}{2} + \frac{x - 2}{4} + \frac{(x - 2)^2}{8}, \]

\[ P_3(x) = \frac{1}{2} + \frac{x - 2}{4} + \frac{(x - 2)^2}{8} + \frac{3(x - 2)^3}{48}. \]

To find the remainder \( R_3(x) \), we have

\[ R_3(x) = \frac{f^{(4)}(c)}{4!} \cdot (x - 2)^4 \]

\[ = \frac{(-24)(4 - c)^{-5} \cdot (-1)}{24} \cdot (x - 2)^4 \]

\[ = \frac{(x - 2)^4}{(4 - c)^5}. \]